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COMPARATIVE SIMULATION STUDY OF LIKELIHOOD RATIO TESTS FOR HOMOGENEITY OF THE EXPONENTIAL DISTRIBUTION

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Abstract

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The aim of this paper is to present and discuss the power of the exact likelihood ratio homogeneity testing procedure of the number of components k in the exponential mixture. First we present the likelihood ratio test for homogeneity (ELR), the likelihood ratio test for homogeneity against two-component exponential mixture (ELR2), and finally the likelihood ratio test for homogeneity against three-component exponential mixture (ELR3). Comparative power study of mentioned homogeneity tests against three-component subpopulation alternative is provided. Therein we concentrate on various setups of the scales and weights, which allow us to make conclusions for generic settings. The natural property is observed, namely increase of the power of exact likelihood ratio ELR, ELR2 and ELR3 tests with scale parameters considered in the alternative. We can state that the differences in power of ELR, ELR2 and ELR3 tests are small – therefore using of the computationally simpler ELR2 test is recommended for broad usage rather than computationally more expensive ELR3 test in the cases when unobserved heterogeneity is modelled. Anyhow caution should be taken before automatic usage of ELR3 in more informative settings, since the application of automatic methods hoping that the data will enforce its true structure is deceptive. Application of obtained results in reliability, finance or social sciences is straightforward.

exponential distribution, homogeneity testing, likelihood ratio, mixture models, Monte Carlo simulations, power study

Currently, many homogeneity tests exist – see Stehlík and Wagner (2012) and references therein. In this paper we focus primarily on likelihood ratio tests. The exact likelihood ratio test for scale and homogeneity in the complete sample from gamma family was derived in Stehlík (2003). The exact distribution of the likelihood ratio test for homogeneity was derived in Stehlík (2006) for the exponential and Weibull distribution and for the generalized gamma distribution was derived in Stehlík (2008). Exact likelihood testing for homogeneity of the number of components in the Rayleigh mixture for $k = 2$ and $k = 3$ components was introduced in Stehlík and Ososkov (2003), and for $k = 2$ in exponential mixture was studied by Stehlík and Wagner (2011), and finally for $k = 3$ in the Rayleigh family was studied in Střelec and Stehlík (2012).

The aim of this paper is to present and discuss the power of the likelihood ratio homogeneity testing procedure for the number of components k in the exponential mixture. In other words, the aim of this paper is to present and discuss the power of exact procedure for testing exponential homogeneity against alternatives of exponential heterogeneity. Therefore, the rest of this paper is organized as follows. In this section, the exact likelihood ratio homogeneity tests are introduced. In Section 2, alternatives to homogeneity and simulations setup are specified. In Section 3, a comparative power study of exact likelihood ratio tests for homogeneity against the three-component subpopulation alternative is provided. Last sections are Conclusions and Summary.

Exact likelihood ratio tests for homogeneity

Firstly, we present exact likelihood ratio tests for homogeneity used for comparative power study.

ELR test

Let $y_1,\ldots y_N$ be independently distributed with exponential densities with unknown scale parameter θ. Then following Stehlík (2006, Theorem 3), the ELR test statistic $-\ln\lambda_N(y)$, where $\lambda_N(y)$ is the formula for likelihood ratio, has the following form

$$-\ln\lambda_N(y) = N\ln\left(\sum_{i=1}^{N} y_i\right) - N\ln N - \sum_{i=1}^{N}\ln y_i \,. \qquad (1)$$

The ELR test statistic has some important properties – e.g. scale invariance, i.e. the distribution of the test statistic under null hypothesis is independent of the unknown scale parameter (Stehlík, 2006) and it is optimal in the Bahadur sense (see Rublík, 1989a, 1989b).

ELR2 test

ELR2 is test constructed for testing of homogeneity of the number of components k in mixture for $k = 2$ components, firstly introduced by Stehlík and Ososkov (2003). Therefore, we consider the testing problem of the form

$$H_0 : k = 1 \text{ vs. } H_1 : k = 2, \qquad (2)$$

which can be, following Stehlík and Ososkov (2003), in the mixture model approximated by the hypothesis of the subpopulation model

$$H_0 : \theta_1 = \ldots \theta_N \text{ vs. } H_1 : \exists \text{ non empty disjoint subsets } M_1, M_2, M_1 \cup M_2 = \{1, \ldots, N\}, \qquad (3)$$

where $M_1 \cap M_2 = \emptyset$, M_1, $M_2 \neq \emptyset$, $\forall_j \in M_1 : \theta_j = \theta_1$, $\forall_j \in M_2 : \theta_j = \theta_2$, where θ_1 and θ_2 are different scale parameters, i.e. $\theta_1 \neq \theta_2$.

Following Stehlík and Ososkov (2003) and Stehlík and Wagner (2011) we introduce the exact likelihood ratio test of the hypothesis (3) which approximates the hypothesis (2). Let $y_1,\ldots y_N$ be independently distributed with exponential densities and suppose that $\{y_{i_1}, \ldots, y_{i_K}\}$, $0 < K < N$ are the observations from exponential distribution with scale parameter θ_1 and the other observations are distributed according to the exponential distribution with scale parameter θ_2 and where i_k denotes indices from set $\{1,\ldots,N\}$ for $1 \leq i \leq K$. Then following Stehlík and Ososkov (2003), the formula for likelihood ratio has the following form

$$\lambda_N(y) = \min_{0<K<N,\, p\in P(K)} \left\{ \frac{N^N}{K^K (N-K)^{N-K}} \cdot \frac{\left(y_{i_1} + \ldots + y_{i_K}\right)^K \left(y_{i_{K+1}} + \ldots + y_{i_N}\right)^{N-K}}{\left(y_1 + \ldots + y_N\right)^N} \right\}, \qquad (4)$$

where $P(K)$ for $0 < K < N$ denotes all partitions of $\{1,\ldots,K\}$ in two non-empty subsets.

Then ELR2 test statistic $-\ln\lambda_N(y)$, where $\lambda_N(y)$ is given by formula (4), has the following form

$$-\ln\lambda_N(y) = -\min_{0<K<N,\, p\in P(K)} \left\{ \begin{array}{l} N\ln N - K\ln K - (N-K)\ln(N-K) + K\ln\left(\sum_{n=1}^{K} y_{i_n}\right) + \\ +(N-K)\ln\left(\sum_{n=1}^{N-K} y_{i_n}\right) - N\ln\left(\sum_{n=1}^{N} y_n\right) \end{array} \right\}. \qquad (5)$$

The ELR2 test statistic is scale invariant under the null hypothesis (see Stehlík and Ososkov, 2003). Following Stehlík and Wagner (2011, Lemma 3.1), ELR2 test statistic can be also determined as

$$\ln\lambda_N(y) = N\ln N - N\ln\left(\sum_{n=1}^{N} y_n\right) + H_{\min}\,, \qquad (6)$$

where $H_{\min}$ can be determined as sums of order statistics $y_{(i)}$, i.e.

$$H_{\min} = \min_{0<K<N} \left\{ -K\ln K - (N-K)\ln(N-K) + K\ln\left(\sum_{i=1}^{K} y_{(i)}\right) + (N-K)\ln\left(\sum_{i=K+1}^{N} y_{(i)}\right) \right\}. \qquad (7)$$

ELR3 test

ELR3 is test constructed for testing of homogeneity of the number of components k in mixture for $k = 3$ components, similarly as ELR2 test introduced by Stehlík and Ososkov (2003). Therefore, we consider the testing problem of the form

$$H_0 : k = 1 \text{ vs. } H_1 : k = 3 \qquad (8)$$

which can be, following Stehlík and Ososkov (2003), in the mixture model approximated by the hypothesis of the subpopulation model

$$H_0 : \theta_1 = \ldots \theta_N \text{ vs. } H_1 : \exists \text{ non empty disjoint subsets } M_1, M_2, M_3 \qquad (9)$$

of the set $1,\ldots,N$ such that $\forall_j \in M_1 : \theta_j = \theta_1$, $\forall_j \in M_2 : \theta_j = \theta_2$, $\forall_j \in M_3 : \theta_j = \theta_3$, where θ_1, θ_2, and θ_3 are different scale parameters.

Following Stehlík and Ososkov (2003) we introduce the exact likelihood ratio test of the hypothesis (9) which approximates the hypothesis (8). Let $y_1,\ldots y_N$ be independently distributed with exponential densities and suppose that $\{y_{i_1},\ldots,y_{i_K}\}$, $0 < K < N - 1$ are the observations from exponential distribution with scale parameter θ_1, $\{y_{j_1},\ldots,y_{j_L}\}$, $0 < L < N - K$ are the observations from exponential distribution with scale parameter θ_2, and finally the other observations are distributed according to the exponential distribution with scale parameter θ_3 and where i_k denotes indices from set $\{1,\ldots,N\}$ for $1 \leq i \leq K$ and j_l denotes indices from set $\{1,\ldots,N\}$ for $1 \leq j \leq L$. Then following Stehlík and Ososkov (2003), the formula for likelihood ratio has the following form

$$\lambda_N(y) = \min_{\substack{0<K<N-1,\\ 0<L<N-K,\\ p\in P(K,L)}} \left\{ \begin{array}{l} \dfrac{N^N}{K^K L^L (N-K-L)^{N-K-L}} \cdot \\ \cdot \dfrac{\left(y_{i_1}+\ldots+y_{i_K}\right)^K \left(y_{j_1}+\ldots+y_{j_L}\right)^L \left(y_{l_1}+\ldots+y_{l_{N-K-L}}\right)^{N-K-L}}{\left(y_1+\ldots+y_N\right)^N} \end{array} \right\}, \quad (10)$$

where $P(K,L)$ for $0 < K < N - 1$ and $0 < L < N - K$ denotes all disjoint pairs of K-subsets $\{y_{i_1},\ldots,y_{i_K}\}$ and L-subsets $\{y_{j_1},\ldots,y_{j_L}\}$ of the set $\{1,\ldots,N\}$, where i_k denotes indices from set $\{1,\ldots,N\}$ for $1 \le i \le K$ and j_l denotes indices from set $\{1,\ldots,N\}$ for $1 \le j \le L$. Then ELR3 test statistic $-\ln\lambda_N(y)$, where $\lambda_N(y)$ is given by formula (10), has the following form

$$-\ln\lambda_N(y) = -\min_{\substack{0<K<N-1,\\ 0<L<N-K,\\ p\in P(K,L)}} \left\{ \begin{array}{l} N\ln N - K\ln K - L\ln L - (N-K-L)\ln(N-K-L) + \\ +K\ln\left(\sum_{n=1}^{K} y_{i_n}\right) + L\ln\left(\sum_{n=1}^{L} y_{j_n}\right) + (N-K-L)\ln\left(\sum_{n=1}^{N-K-L} y_{l_n}\right) + \\ -N\ln\left(\sum_{n=1}^{N} y_n\right) \end{array} \right\}. \quad (11)$$

MATERIALS AND METHODS

Alternatives to homogeneity

As Stehlík and Wagner (2011) state, alternatives to homogeneity are often specified as mixture models and the most popular alternative to homogeneity is the mixture model with exponential components. The joint density of a sample $y_1,\ldots y_N$ from a general k-component mixture of exponential components is

$$f(y_1,\ldots,y_N) = \prod_{i=1}^{N}\left(\sum_{j=1}^{k} p_j\theta_j \exp\left(-\theta_j y_i\right)\right), \quad (12)$$

where $0 < p_j < 1, \sum_j p_j = 1$.

In this paper we will present and discuss the power of the exact likelihood ratio homogeneity testing procedure of the number of components k in the exponential mixture only for $k = 3$ components, introduced by Stehlík and Ososkov (2003). Therefore we suppose following hypothesis

$$H_0: y_1, \ldots, y_N \sim \text{Exponential}(\theta)$$

$$\text{vs.} \quad (13)$$

$$H_1: y_1, \ldots, y_N \text{ follow a mixture of three exponential components,}$$

i.e. we suppose mixture of three exponential components with following probability density function (pdf)

$$f(y) = p_K\theta_1\exp(-\theta_1 y) + p_L\theta_2\exp(-\theta_2 y) + (1-p_K-p_L)\theta_3\exp(-\theta_3 y), \quad (14)$$

where p_K, p_L and $1 - p_K - p_L$ are weights of components such that $0 < p_K, p_L, 1 - p_K - p_L < 1$.

Setup of simulation study

A simulation study was performed to compare the power of the exact likelihood ratio tests ELR, ELR2 and ELR3 for the following parameters setup: $N \in \{10, 20, 50\}$, $\theta_1 = 1$, $\theta_2 \in \{1, 3, 5, 7\}$ and $\theta_3 \in \{1, 5, 7, 10\}$ and different component weights $p_K \in \{0.2, 0.4, 0.5, 0.6, 0.8\}$ and $p_L \in \{0.1, 0.2, 0.3, 0.4, 0.6\}$. For mentioned parameters setup $M = 10000$ samples were generated and the proportion of rejections of ELR, ELR2 and ELR3 tests was determined. Note that the ELR, ELR2 and ELR3 tests have non standard asymptotic distributions but we can simulate their exact distributions. Therefore, critical values of the ELR, ELR2 and ELR3 tests can be simply obtained by Monte Carlo simulations, i.e. we generated $M = 100000$ samples of size $N \in \{10, 20, 50\}$ from the standard exponential distribution, then we computed the test statistic for each sample, and finally critical values $c_{1-\alpha}$ were determined.

RESULTS AND DISCUSSION

Critical values of the ELR, ELR2 and ELR3 test statistics for $\alpha \in \{0.01, 0.05, 0.10\}$ based on $M = 100000$ samples of size $N \in \{10, 20, 50\}$ from the standard exponential distribution are presented in Tab. I.

Consequently, sizes of the ELR, ELR2 and ELR3 test statistics for $\alpha = 0.05$ (i.e. power of the ELR, ELR2 and ELR3 test statistics against mixture of three exponential components with pdf from (14) for $\theta_1 = \theta_2 = \theta_3 = 1$) are presented in Tab. II. As we can see from Tab. II, the ELR, ELR2 and ELR3 tests hold the chosen size $\alpha = 0.05$ even for small samples. The biggest difference between theoretical and empirical sizes is for ELR3 test for $N = 50$ and $p_K = 0.50$ and $p_L = 0.20$ – for this parameters setup the difference between theoretical and empirical sizes is 0.006.

Finally, power of exact likelihood ratio tests ELR, ELR2 and ELR3 against mixture of three exponential components with probability density

I: *Critical values of ELR, ELR2 and ELR3 test statistics*

	$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.10$		
	$N = 10$	$N = 20$	$N = 50$	$N = 10$	$N = 20$	$N = 50$	$N = 10$	$N = 20$	$N = 50$
ELR	12.3471	20.7345	42.8979	9.7898	17.3753	38.1566	8.5527	15.7070	35.7611
ELR2	8.7502	13.4598	26.2985	6.8449	11.1932	23.2229	5.9695	10.0873	21.6639
ELR3	11.0170	17.3062	34.0744	8.7134	14.5408	30.3562	7.6242	13.1624	28.4471

Source: own simulations

II: *Size of the ELR, ELR2 and ELR3 tests for* $\alpha = 0.05$

	$\theta_1 = \theta_2 = \theta_3 = 1$							
	p_K, p_L	$N = 10$	$N = 20$	$N = 50$	p_K, p_L	$N = 10$	$N = 20$	$N = 50$
ELR	$p_K = 0.20$ $p_L = 0.20$	0.051	0.053	0.050	$p_K = 0.50$ $p_L = 0.30$	0.051	0.048	0.051
ELR2		0.051	0.050	0.051		0.052	0.046	0.054
ELR3		0.051	0.053	0.051		0.051	0.047	0.051
ELR	$p_K = 0.20$ $p_L = 0.40$	0.048	0.052	0.047	$p_K = 0.50$ $p_L = 0.40$	0.051	0.048	0.048
ELR2		0.049	0.052	0.048		0.052	0.050	0.049
ELR3		0.048	0.051	0.046		0.053	0.049	0.048
ELR	$p_K = 0.20$ $p_L = 0.60$	0.048	0.049	0.050	$p_K = 0.60$ $p_L = 0.10$	0.054	0.046	0.052
ELR2		0.051	0.048	0.052		0.054	0.048	0.052
ELR3		0.048	0.048	0.050		0.054	0.046	0.051
ELR	$p_K = 0.40$ $p_L = 0.20$	0.049	0.051	0.051	$p_K = 0.60$ $p_L = 0.20$	0.051	0.053	0.051
ELR2		0.050	0.050	0.047		0.052	0.052	0.050
ELR3		0.048	0.050	0.050		0.051	0.052	0.049
ELR	$p_K = 0.40$ $p_L = 0.40$	0.051	0.051	0.051	$p_K = 0.60$ $p_L = 0.30$	0.050	0.050	0.050
ELR2		0.050	0.046	0.052		0.049	0.048	0.052
ELR3		0.050	0.050	0.050		0.050	0.049	0.050
ELR	$p_K = 0.50$ $p_L = 0.10$	0.049	0.049	0.050	$p_K = 0.80$ $p_L = 0.10$	0.051	0.051	0.054
ELR2		0.046	0.049	0.048		0.052	0.049	0.052
ELR3		0.049	0.049	0.050		0.052	0.050	0.054
ELR	$p_K = 0.50$ $p_L = 0.20$	0.051	0.048	0.053				
ELR2		0.050	0.048	0.054				
ELR3		0.052	0.049	0.056				

function (14) for parameters setup mentioned above is reported in Tab. III–V. Firstly, Tab. III presents power of ELR, ELR2 and ELR3 tests against mixture of three exponential components for $\theta_1 = 1$, $\theta_2 = 3$, $\theta_3 = 5$ and $\theta_1 = 1$, $\theta_2 = 3$, $\theta_3 = 7$. In Tab. IV power of ELR, ELR2 and ELR3 tests against mixture of three exponential components for $\theta_1 = 1$, $\theta_2 = 3$, $\theta_3 = 10$ and $\theta_1 = 1$, $\theta_2 = 5$, $\theta_3 = 7$ is presented. Finally, last Tab. V presents power of ELR, ELR2 and ELR3 tests against mixture of three exponential components for $\theta_1 = 1$, $\theta_2 = 5$, $\theta_3 = 10$ and $\theta_1 = 1$, $\theta_2 = 7$, $\theta_3 = 10$.

As it can be seen from Tab. III–V, the power of the ELR, ELR2 and ELR3 tests increases with scale parameters $\theta = (\theta_2, \theta_3)$. For example the power of the most powerful ELR3 test against mixture of three exponential components for $\theta_1 = 1$, $\theta_2 = 3$, $N = 50$ and component weights $p_K = 0.20$ and $p_L = 0.40$ is 0.478 for $\theta_3 = 5$, 0.684 for $\theta_3 = 7$, and finally 0.861 for $\theta_3 = 10$.

For fixed θ the highest power is obtained for component weights $p_K = 0.50$, $p_L = 0.10$ and $\theta_2 < \theta_3$. On the other hand the lowest power is obtained for high component weight $p_K = 0.80$, which means that first component of mixture of three exponential components is predominant and this lower contamination is for ELR, ELR2 and ELR3 tests hard to detect.

The differences in power of ELR, ELR2 and ELR3 tests for fixed θ, N and component weights p_K and p_L are small – the highest difference in power of mentioned tests is against mixture of three exponential components for $\theta_1 = 1$, $\theta_2 = 3$, $\theta_3 = 7$, $N = 50$ and component weights $p_K = 0.60$ and $p_L = 0.20$ where power of ELR test is 0.611 and power of the most powerful ELR2 test is 0.662, i.e. difference between the most powerful ELR2 test and the less powerful ELR test is 0.051.

Small differences between the most and less powerful tests mean that powers of analyzed ELR, ELR2 and ELR3 tests are comparable. It can be also demonstrated by following percent proportion of cases in which analyzed tests show the highest power – as it is evident from Tab. III–V, the ELR test shows the highest power in 3.6 percent of analyzed cases, the ELR2 test in 58.8 percent of analyzed cases, and finally the ELR3 shows the highest power in 37.6 percent of analyzed cases. Based on simulation results reported above, we can also state that the ELR3 test outperforms the ELR and ELR2 tests in most cases for $p_K < 0.40$, while the ELR2 test outperforms the ELR and ELR3 tests for $p_K \geq 0.40$.

CONCLUSIONS

Using of exponential distribution is broad – e.g. the exponential distribution is one of the most widely used lifetime distribution in reliability engineering (see Stehlík and Wagner, 2011). Homogeneity testing provided in this paper can be also of importance for studying of mixed risks in portfolio (see e.g. Potocký, 2008).

As can be seen from results presented above, the power of analyzed ELR, ELR2 and ELR3 tests is

III: *Power of the ELR, ELR2 and ELR3 tests against mixture of three exponential components for* $\alpha = 0.05, \theta_1 = 1, \theta_2 = 3, \theta_3 = 5$ *and* $\theta_1 = 1, \theta_2 = 3, \theta_3 = 7$

p_K, p_L	test	$\theta_1 = 1, \theta_2 = 3, \theta_3 = 5$			$\theta_1 = 1, \theta_2 = 3, \theta_3 = 7$		
		$N = 10$	$N = 20$	$N = 50$	$N = 10$	$N = 20$	$N = 50$
$p_K = 0.20$ $p_L = 0.20$	ELR	**0.212**	0.325	0.565	0.303	0.479	0.790
	ELR2	0.205	0.314	0.541	0.299	0.476	0.783
	ELR3	**0.212**	**0.328**	**0.568**	**0.305**	**0.488**	**0.803**
$p_K = 0.20$ $p_L = 0.40$	ELR	**0.188**	**0.269**	0.474	0.243	0.386	0.667
	ELR2	0.179	0.260	0.452	0.237	0.378	0.669
	ELR3	**0.188**	0.268	**0.478**	**0.245**	**0.392**	**0.684**
$p_K = 0.20$ $p_L = 0.60$	ELR	**0.149**	0.214	**0.372**	**0.180**	**0.269**	0.469
	ELR2	0.144	0.204	0.347	0.172	0.256	0.465
	ELR3	**0.149**	**0.215**	0.371	0.178	**0.269**	**0.478**
$p_K = 0.40$ $p_L = 0.20$	ELR	0.231	0.350	0.636	0.321	0.509	0.831
	ELR2	**0.233**	0.357	0.657	**0.325**	**0.529**	0.856
	ELR3	0.232	**0.363**	**0.663**	0.324	0.527	**0.860**
$p_K = 0.40$ $p_L = 0.40$	ELR	0.184	0.277	0.496	**0.222**	0.350	0.622
	ELR2	0.181	0.277	0.503	0.216	0.359	0.644
	ELR3	**0.185**	**0.283**	**0.516**	0.221	**0.363**	**0.649**
$p_K = 0.50$ $p_L = 0.10$	ELR	0.236	0.366	0.665	0.320	0.525	0.865
	ELR2	**0.239**	**0.379**	**0.695**	**0.333**	**0.563**	**0.893**
	ELR3	0.238	0.376	0.694	0.326	0.545	0.890
$p_K = 0.50$ $p_L = 0.20$	ELR	0.202	0.315	0.586	0.280	0.433	0.771
	ELR2	**0.203**	0.325	0.615	0.283	**0.455**	**0.803**
	ELR3	**0.203**	**0.328**	**0.619**	**0.284**	0.450	0.799
$p_K = 0.50$ $p_L = 0.30$	ELR	0.178	0.266	0.495	0.219	0.338	0.638
	ELR2	0.175	**0.279**	**0.523**	**0.226**	**0.354**	0.668
	ELR3	**0.179**	0.276	0.522	0.221	0.349	**0.669**
$p_K = 0.50$ $p_L = 0.40$	ELR	0.158	0.223	0.405	**0.171**	0.265	0.480
	ELR2	0.157	0.223	0.423	0.170	**0.272**	0.496
	ELR3	**0.160**	**0.229**	**0.431**	0.170	0.271	**0.505**
$p_K = 0.60$ $p_L = 0.10$	ELR	0.194	0.306	0.559	0.264	0.431	0.760
	ELR2	0.193	**0.324**	**0.596**	**0.277**	**0.470**	**0.808**
	ELR3	**0.197**	0.321	0.591	0.267	0.455	0.801
$p_K = 0.60$ $p_L = 0.20$	ELR	0.167	0.260	0.463	0.203	0.322	0.611
	ELR2	0.169	**0.273**	**0.497**	**0.210**	**0.345**	**0.662**
	ELR3	**0.170**	0.270	0.489	0.207	0.338	0.648
$p_K = \mathbf{0.60}$ $p_L = \mathbf{0.30}$	ELR	0.142	0.207	0.368	0.167	0.237	0.440
	ELR2	**0.143**	0.212	**0.387**	**0.171**	**0.251**	0.467
	ELR3	0.142	**0.213**	**0.387**	0.168	0.246	**0.468**
$p_K = \mathbf{0.80}$ $p_L = \mathbf{0.10}$	ELR	0.102	0.138	0.235	0.120	0.173	0.295
	ELR2	**0.107**	**0.146**	**0.254**	0.120	**0.181**	**0.326**
	ELR3	0.105	0.140	0.249	**0.122**	0.180	0.317

comparable, i.e. differences in power of analyzed tests are very small – max. difference in power is 0.051 between ELR2 and ELR tests. Similarly, the ELR2 test shows the highest power in almost 59 percent of analyzed cases and the ELR3 test in almost 38 percent of analyzed cases. These comparable results mean that using of the computationally simpler ELR2 test is recommended for broad usage rather than computationally more expensive ELR3 test, especially for large sample sizes.

IV: *Power of the ELR, ELR2 and ELR3 tests against mixture of three exponential components for* $\alpha = 0.05, \theta_1 = 1, \theta_2 = 3, \theta_3 = 10$ *and* $\theta_1 = 1, \theta_2 = 5, \theta_3 = 7$

p_K, p_L	test	$\theta_1 = 1, \theta_2 = 3, \theta_3 = 10$			$\theta_1 = 1, \theta_2 = 5, \theta_3 = 7$		
		N = 10	**N = 20**	**N = 50**	**N = 10**	**N = 20**	**N = 50**
$p_K = 0.20$ $p_L = 0.20$	**ELR**	0.433	0.667	0.934	**0.330**	0.511	0.803
	ELR2	0.437	0.680	0.940	0.319	0.496	0.793
	ELR3	**0.439**	**0.685**	**0.943**	**0.330**	**0.518**	**0.814**
$p_K = 0.20$ $p_L = 0.40$	**ELR**	0.336	0.538	0.843	0.295	0.464	0.763
	ELR2	0.333	0.547	0.859	0.286	0.453	0.746
	ELR3	**0.337**	**0.553**	**0.861**	**0.297**	**0.469**	**0.772**
$p_K = 0.20$ $p_L = 0.60$	**ELR**	**0.225**	0.337	0.617	0.270	0.412	0.694
	ELR2	0.214	0.332	0.622	0.260	0.399	0.678
	ELR3	**0.225**	**0.338**	**0.633**	**0.272**	**0.419**	**0.706**
$p_K = 0.40$ $p_L = 0.20$	**ELR**	0.427	0.677	0.948	0.373	0.585	0.894
	ELR2	**0.445**	**0.711**	**0.966**	**0.381**	**0.607**	0.910
	ELR3	0.435	0.700	0.963	0.378	**0.607**	**0.913**
$p_K = 0.40$ $p_L = 0.40$	**ELR**	0.266	0.429	0.761	0.327	0.512	0.832
	ELR2	**0.268**	**0.446**	0.783	**0.335**	**0.529**	0.852
	ELR3	**0.268**	0.443	**0.787**	0.334	**0.529**	**0.857**
$p_K = 0.50$ $p_L = 0.10$	**ELR**	0.469	0.710	0.970	0.362	0.576	0.902
	ELR2	**0.488**	**0.754**	**0.983**	**0.376**	**0.619**	**0.926**
	ELR3	0.476	0.734	0.979	0.368	0.602	0.923
$p_K = 0.50$ $p_L = 0.20$	**ELR**	0.362	0.587	0.904	0.339	0.532	0.867
	ELR2	**0.375**	**0.618**	**0.927**	**0.352**	**0.568**	**0.895**
	ELR3	0.367	0.608	0.924	0.347	0.559	0.894
$p_K = 0.50$ $p_L = 0.30$	**ELR**	0.269	0.446	0.776	0.309	0.482	0.832
	ELR2	**0.274**	**0.470**	**0.803**	**0.318**	**0.517**	**0.863**
	ELR3	0.271	0.462	**0.803**	0.315	0.505	0.861
$p_K = 0.50$ $p_L = 0.40$	**ELR**	0.202	0.295	0.570	0.294	0.454	0.782
	ELR2	0.202	0.299	0.589	0.298	**0.477**	**0.815**
	ELR3	**0.203**	**0.303**	**0.596**	**0.299**	0.472	**0.815**
$p_K = 0.60$ $p_L = 0.10$	**ELR**	0.360	0.594	0.916	0.304	0.482	0.826
	ELR2	**0.376**	**0.636**	**0.945**	**0.319**	**0.517**	**0.872**
	ELR3	0.364	0.620	0.938	0.312	0.508	0.858
$p_K = 0.60$ $p_L = 0.20$	**ELR**	0.259	0.437	0.771	0.274	0.447	0.766
	ELR2	**0.266**	**0.458**	**0.805**	**0.290**	**0.478**	**0.815**
	ELR3	0.261	0.451	0.799	0.279	0.464	0.806
$p_K = 0.60$ $p_L = 0.30$	**ELR**	0.187	0.290	0.535	0.242	0.388	0.720
	ELR2	**0.190**	0.299	0.559	**0.255**	**0.418**	**0.768**
	ELR3	0.188	**0.301**	**0.562**	0.247	0.410	0.759
$p_K = 0.80$ $p_L = 0.10$	**ELR**	**0.147**	0.210	0.395	0.144	0.217	0.406
	ELR2	**0.147**	**0.218**	0.415	**0.149**	**0.234**	**0.447**
	ELR3	0.146	0.215	**0.417**	**0.149**	0.226	0.432

SUMMARY

In this study, power of three likelihood ratio tests for homogeneity testing procedure of the number of components k in the exponential mixture for $k = 3$, firstly introduced by Stehlík and Ososkov (2003). Therefore, in this study, we presented and discussed the power of the exact likelihood ratio tests ELR, ELR2 and ELR3 against three-component subpopulation alternative – i.e. mixture of three exponential components for various parameters setup. For purpose of power comparison $M = 10000$ samples were generated and the proportion of rejections of ELR, ELR2 and ELR3 tests was determined.

V: *Power of the ELR, ELR2 and ELR3 tests against mixture of three exponential components for* $\alpha = 0.05, \theta_1 = 1, \theta_2 = 5, \theta_3 = 10$ *and* $\theta_1 = 1, \theta_2 = 7, \theta_3 = 10$

p_K, p_L	test	$\theta_1 = 1, \theta_2 = 5, \theta_3 = 10$			$\theta_1 = 1, \theta_2 = 7, \theta_3 = 10$		
		N = 10	**N = 20**	**N = 50**	**N = 10**	**N = 20**	**N = 50**
$p_K = 0.20$ $p_L = 0.20$	**ELR**	0.447	0.660	0.934	0.477	0.686	0.945
	ELR2	0.443	0.665	0.933	0.470	0.686	0.941
	ELR3	**0.452**	**0.674**	**0.944**	**0.482**	**0.697**	**0.951**
$p_K = 0.20$ $p_L = 0.40$	**ELR**	0.384	0.593	0.893	0.428	0.652	0.919
	ELR2	0.384	0.592	0.888	0.425	0.645	0.916
	ELR3	**0.389**	**0.605**	**0.897**	**0.436**	**0.661**	**0.926**
$p_K = 0.20$ $p_L = 0.60$	**ELR**	0.308	0.483	0.786	0.389	0.585	0.894
	ELR2	0.301	0.477	0.775	0.382	0.575	0.880
	ELR3	**0.310**	**0.489**	**0.797**	**0.394**	**0.595**	**0.901**
$p_K = 0.40$ $p_L = 0.20$	**ELR**	0.480	0.741	0.977	0.537	0.780	0.984
	ELR2	**0.500**	**0.767**	0.983	**0.552**	**0.811**	**0.989**
	ELR3	0.493	0.762	**0.984**	0.547	0.804	**0.989**
$p_K = 0.40$ $p_L = 0.40$	**ELR**	0.389	0.600	0.908	0.479	0.717	0.966
	ELR2	**0.395**	**0.622**	0.922	**0.493**	**0.747**	**0.975**
	ELR3	0.394	0.618	**0.925**	0.490	0.739	0.974
$p_K = 0.50$ $p_L = 0.10$	**ELR**	0.486	0.744	0.980	0.509	0.775	0.982
	ELR2	**0.509**	**0.789**	**0.987**	**0.542**	**0.814**	**0.991**
	ELR3	0.499	0.771	**0.987**	0.523	0.799	0.990
$p_K = 0.50$ $p_L = 0.20$	**ELR**	0.429	0.671	0.949	0.479	0.734	0.974
	ELR2	**0.446**	**0.709**	**0.967**	**0.506**	**0.777**	**0.985**
	ELR3	0.437	0.698	0.964	0.491	0.758	0.983
$p_K = 0.50$ $p_L = 0.30$	**ELR**	0.363	0.591	0.907	0.451	0.704	0.965
	ELR2	**0.379**	**0.622**	**0.929**	**0.473**	**0.742**	**0.975**
	ELR3	0.372	0.612	**0.929**	0.460	0.730	0.974
$p_K = 0.50$ $p_L = 0.40$	**ELR**	0.314	0.504	0.833	0.420	0.663	0.945
	ELR2	**0.322**	**0.536**	**0.865**	**0.444**	**0.695**	**0.963**
	ELR3	0.321	0.526	0.861	0.430	0.686	0.960
$p_K = 0.60$ $p_L = 0.10$	**ELR**	0.389	0.635	0.944	0.437	0.686	0.961
	ELR2	**0.414**	**0.684**	**0.967**	**0.462**	**0.732**	**0.981**
	ELR3	0.400	0.665	0.957	0.445	0.711	0.975
$p_K = 0.60$ $p_L = 0.20$	**ELR**	0.343	0.538	0.879	0.392	0.628	0.940
	ELR2	**0.353**	**0.583**	**0.914**	**0.417**	**0.678**	**0.964**
	ELR3	0.343	0.566	0.905	0.403	0.656	0.958
$p_K = 0.60$ $p_L = 0.30$	**ELR**	0.278	0.449	0.783	0.372	0.582	0.909
	ELR2	**0.280**	**0.479**	**0.825**	**0.389**	**0.627**	**0.937**
	ELR3	**0.280**	0.469	0.818	0.378	0.603	0.931
$p_K = 0.80$ $p_L = 0.10$	**ELR**	0.171	0.260	0.514	0.187	0.303	0.588
	ELR2	**0.179**	**0.278**	**0.547**	**0.200**	**0.328**	**0.636**
	ELR3	0.171	0.269	0.538	0.189	0.318	0.621

As can be seen from results of Monte Carlo simulations the power of the exact likelihood ratio ELR, ELR2 and ELR3 tests increases with scale parameters $\theta = (\theta_2, \theta_3)$. We found that for fixed θ the highest power is obtained for component weights $p_K = 0.50$, $p_L = 0.10$ and $\theta_2 < \theta_3$. On the other hand the lowest power is obtained for high component weight $p_K = 0.80$.
But for fixed θ, N and component weights p_K and p_L used in this paper we can state that the differences in power of ELR, ELR2 and ELR3 tests are small – therefore using of the computationally simpler ELR2 test is recommended for broad usage rather than computationally more expensive ELR3 test.

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