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A NOTE ON IMPERFECT HEDGING: A METHOD FOR TESTING STABILITY OF THE HEDGE RATIO

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Abstract

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Companies producing, processing and consuming commodities in the production process often hedge their commodity expositions using derivative strategies based on different, highly correlated underlying commodities. Once the open position in a commodity is hedged using a derivative position with another underlying commodity, the appropriate hedge ratio must be determined in order the hedge relationship be as effective as possible. However, it is questionable whether the hedge ratio determined at the inception of the risk management strategy remains stable over the whole period for which the hedging strategy exists. Usually it is assumed that in the short run, the relationship (say, correlation) between the two commodities remains stable, while in the long run it may vary. We propose a method, based on statistical theory of stability, for on-line detection whether market movements of prices of the commodities involved in the hedge relationship indicate that the hedge ratio may have been subject to a recent change. The change in the hedge ratio decreases the effectiveness of the original hedge relationship and creates a new open position. The method proposed should inform the risk manager that it could be reasonable to adjust the derivative strategy in a way reflecting the market conditions after the change in the hedge ratio.

stability analysis, imperfect hedging, changepoint detection

1 INTRODUCTION

Companies producing, consuming, processing or storing commodities often face the risk of changes in market prices. They implement various hedging strategies to minimize that risk. For example, airline companies often need to hedge their future purchases of kerosene; producers of agricultural commodities need to hedge their future sales; companies processing metals need to hedge the metal held on stock.

In this text we will deal with the example of an airline company wishing to hedge a future purchase of one tone kerosene. The company uses forwards/futures the underling variable of which is different, but highly correlated with kerosene. As an example we say that the company uses crude oil futures. This situation happens whenever we need to hedge an exposition in a commodity for which derivatives are not available; then it is necessary to choose another commodity, as highly correlated as possible, which is traded at derivative markets.

When such a risk management strategy is introduced, the basic question is: what is the optimal volume of crude oil futures to be bought for hedging the open position in one ton of kerosene? That is, shall the exposition in one ton of kerosene be hedged using the volume of 1.1 t, 1.2 t, 1.3 t or 1.4 t crude oil futures? That ratio is known as *hedge ratio*. At the inception of the risk-management strategy, the hedge ratio is usually estimated from historical data. Of course, a wrong estimate may be expensive: if the hedge ratio is estimated too low, then a certain position in kerosene remains unhedged. On the other hand, if the hedge ratio is estimated too high, then a new open position in crude oil originates.

In order the risk-management strategy make sense it is necessary to assume that there is a stable relationship between the two commodities, say x = crude oil and y = kerosene. Without this assumption, hedging of y using x would not make sense. However, during the life of the strategy it might happen that the assumption is more or less violated. For example it may happen that the correlation between x and y vanishes at all. Then the original risk-management strategy changes into a pair of independent speculative positions in two uncorrelated variables. The main aim of this text is to develop a method for detection of a less serious violation of the assumption, which is more probable to occur in practice. We shall deal with the situation that the hedge ratio changes during the life of the risk-management strategy. We develop a tool for detection of this threat. The tool should suggest the risk manager to adjust the strategy accordingly to minimize the risk arising from the change.

The method is an extension of the approach studied in Černý (2011b) where the problem of estimation of the hedge ratio at the time of inception of the hedging strategy was considered. The approach has been also successful in volatility analysis; see Černý (2008).

The problem of stability of the hedge ratio in time has been also considered, though from a different perspective, in Choudhry (2009), McMillan (2005) and Lien and Shrestha (2008).

2 The relationship between *x* and *y*

We have a risk management strategy where the open position in a variable y (kerosene) is hedged with a futures position in another variable x (crude oil). Assume that a long-term relationship between y and x is in one of the following forms:

$$\Delta y_t = \lambda \Delta x_t + \varepsilon_t, \tag{1a}$$

$$\Delta y_t = \alpha + \lambda \Delta x_t + \varepsilon_t, \tag{1b}$$

$$y_t = \lambda x_t + \varepsilon_t, \tag{1c}$$

$$y_t = \alpha + \lambda x_t + \varepsilon_t, \tag{1d}$$

$$\Delta \log y_t = \lambda + \Delta \log x_t + \varepsilon_t, \tag{1e}$$

where t is the index of time, Δ is the difference operator, ε_t is the random error and α and λ are parameters. (The assumptions on ε_t will be specified later.) Figure 1, where real data are plotted, shows that the assumption of the relationship between y and x of type (1a)–(1e) is indeed reasonable.

The parameter λ in (1a)–(1d) denotes the hedge ratio. In (1e), the hedge ratio takes the form e^{λ}. (The parameter α is irrelevant in our context; though it is sometimes involved in the model, usually to achieve a better goodness-of-fit, its value does not affect the hedge effectiveness.)

Let ν_t be homoskedastic with unit variance. The random errors ε_t are usually assumed in one of the forms

$$\varepsilon_t = \sigma\nu_t,\ \ \varepsilon_t = x_t\sigma\nu_t,\ \ \varepsilon_t = y_t\sigma\nu_t,$$

where $\sigma > 0$ is a parameter. As an example we shall assume the model

$$y_t = \alpha + \lambda x_t + x_t\sigma\nu_t. \tag{2}$$

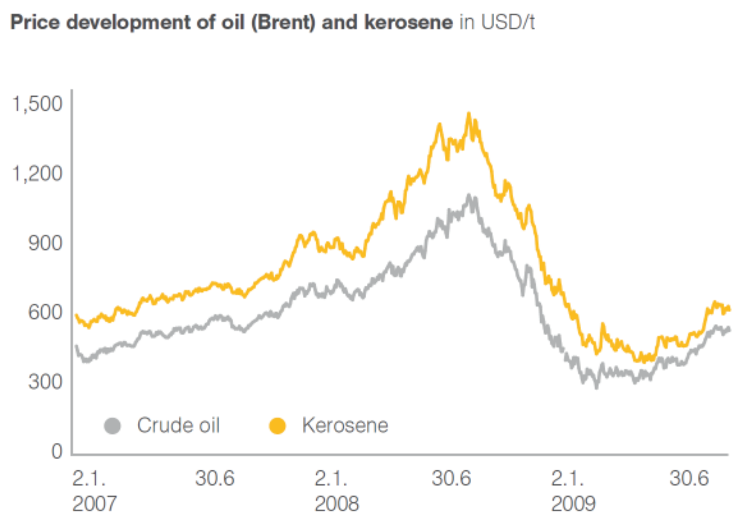
The fact that variance of the error term is proportional to the price level x_t is a traditional feature of financial time series. However, the method of Sections 3–4 could be used for the other discussed models as well.

The hedge ratio λ can be estimated as the absolute term in the homoskedastic model

$$\frac{y_t}{x_t} = \lambda + \alpha\frac{1}{x_t} + \sigma\nu_t \tag{3}$$

which is equivalent to (2).

We shall assume that ν_t's are such that the model (3) can be estimated with Ordinary Least Squares (OLS).



1: *Prices of kerosene and crude oil*
Source: Lufthansa Annual Report

3 A tool for testing stability

Assume that the hedge ratio λ was estimated from historical data using the model (3) at the inception of the risk-management strategy. Let T_0 denote the time of inception of the strategy. Assume further that now we are in time $T^* > T_0$ (but still before maturity of the strategy) and ask a question whether the hedge ratio has changed or not. That is, we ask whether the market data from the period from T_0 up to T^* give evidence that λ estimated at T_0 is no more valid and should be re-estimated.

First we need to derive a statistic for testing the null hypothesis that the ratio remains stable.

Assume that the set of data $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is available. In the derivation of the test, we regard $y_1, \ldots, y_n$ as random variables. Let us test the hypothesis

H: the relationship (3) is valid for all $t \in \{1, \ldots, n\}$

against the alternative

A: there is a time $\tau \in \{3, \ldots, n-3\}$ such that

$$\frac{y_t}{x_t} = \begin{cases} \lambda_0 + \alpha_0 \times \dfrac{1}{x_t} + \sigma\nu_t & \text{for } t \in \{1, \ldots, \tau\}, \\ \lambda_1 + \alpha_1 \times \dfrac{1}{x_t} + \sigma\nu_t & \text{for } t \in \{\tau+1, \ldots, n\}, \end{cases}$$

where $(\alpha_0, \lambda_0) \neq (\alpha_1, \lambda_1)$ and τ are unknown parameters. Assuming that ν_t are $N(0, 1)$ independent, we can construct the log-likelihood ratio

$$\begin{aligned} L &= \ln \frac{f_A}{f_H} \\ &= \ln \prod_{t=1}^{\tau} \frac{1}{\sigma \times \sqrt{2\pi}} \exp\left(-\frac{(y_t - \lambda_0 x_t - \alpha_0)^2}{2x_t^2\sigma^2} \right) \\ &+ \ln \prod_{t=\tau+1}^{n} \frac{1}{\sigma \times \sqrt{2\pi}} \exp\left(-\frac{(y_t - \lambda_1 x_t - \alpha_1)^2}{2x_t^2\sigma^2} \right) \\ &- \ln \prod_{t=1}^{n} \frac{1}{\sigma \times \sqrt{2\pi}} \exp\left(-\frac{(y_t - \lambda x_t - \alpha)^2}{2x_t^2\sigma^2} \right) \\ &= \frac{1}{2\sigma^2}\left[\sum_{t=1}^{n} \frac{(y_t - \lambda x_t - \alpha)^2}{x_t^2} - \sum_{t=1}^{\tau} \frac{(y_t - \lambda_0 x_t - \alpha_0)^2}{x_t^2} - \sum_{t=\tau+1}^{n} \frac{(y_t - \lambda_1 x_t - \alpha_1)^2}{x_t^2} \right], \end{aligned}$$

where f_A and f_H denote the joint distribution of $\frac{y_t}{x_t}$ under *A* and *H*, respectively. If we assume that τ is fixed, we get the log-likelihood test for the existence of change in the regression relationship in time τ of the form

$$U_\tau = \frac{RSS_{1:n} - RSS_{1:\tau} - RSS_{\tau+1:n}}{2\sigma^2},$$

if the standard error σ is known, or

$$U_\tau = \frac{n-2}{2} \times \frac{RSS_{1:n} - RSS_{1:\tau} - RSS_{\tau+1:n}}{RSS_{1:n}},$$

if σ is unknown (which is the most frequent case in practice). The symbol $RSS_{i:j}$ stands for the residual sum of squares from OLS-estimated regression

$$\frac{y_t}{x_t} = \lambda + \alpha \frac{1}{x_t} + \sigma\nu_t$$

using the data set $t \in \{i, i+1, \ldots, j\}$. More precisely, denoting

$$\mathbf{x}_t = (1 \quad \tfrac{1}{x_t}), \quad \mathbf{X}_{i:j} = \begin{pmatrix} \mathbf{x}_i \\ \mathbf{x}_{i+1} \\ \vdots \\ \mathbf{x}_j \end{pmatrix}, \quad \mathbf{z}_{i:j} = \begin{pmatrix} \frac{y_i}{x_i} \\ \frac{y_{i+1}}{x_{i+1}} \\ \vdots \\ \frac{y_j}{x_j} \end{pmatrix},$$

it holds

$$RSS_{i:j} = \| (\mathbf{I} - \mathbf{X}_{i:j}(\mathbf{X}_{i:j}^{\mathrm{T}}\mathbf{X}_{i:j})^{-1}\mathbf{X}_{i:j}^{\mathrm{T}})\mathbf{z}_{i:j} \|^2,$$

where **I** denotes the unit matrix and $|| \cdot ||$ denotes the L_2-norm.

Changing normalization, instead of U_τ we will use an equivalent statistic

$$V_\tau = \frac{RSS_{1:n} - RSS_{1:\tau} - RSS_{\tau+1:n}}{RSS_{1:n}}.$$

The reason for preferring V_τ to U_τ is purely technical and will be apparent later. Relaxing the assumption that τ is fixed we obtain the statistic

$$V = \max_{t \in \{3, \ldots, n-3\}} V_t. \qquad (4)$$

We will also need the statistic *V* applied to a subset $\{i, i+1, \ldots, j\}$ of the set of all observations $\{1, \ldots, n\}$. Thus it will be useful to denote

$$V_{i:j} = \max_{t \in \{i+2, \ldots, j-3\}} V_t^{i:j}, \qquad (5)$$

where

$$V_t^{i:j} = \frac{RSS_{i:j} - RSS_{i:t} - RSS_{t+1:j}}{RSS_{i:j}}.$$

We shall need critical values for the statistic *V* (or $V_{i:j}$) under *H*. The statistic *V*, being the maximum of dependent $B_{1, n/2-2}$-distributed random variables, has a complicated distribution; in fact, an exact formula is not known.

Fortunately, the statistic *V* is essentially the same statistic as investigated by Worsley (1983); this is why we have used V_t instead of U_t in (4). Worsley derived a Bonferroni-type approximation (see also Černý, 2011a) of the distribution of *V* in the form

$$W(z) = \Pr[V \le z] \approx B_{1,\, n/2-2}(z) - \frac{2\beta_{3/2,\, n/2-1}}{\pi(n-2)} \times$$
$$\times \left(\sum_{t=2}^{n-3} \xi_t - \frac{1}{6} \left(\frac{n-5}{6} \times \frac{z}{1-z} - 1 \right) \times \sum_{t=2}^{n-3} \xi_t^3 \right),$$

where

$$\xi_t = \sqrt{\mathbf{x}_{t+1}^{\mathrm{T}} (\mathbf{X}_{t+1:n}^{\mathrm{T}} \mathbf{X}_{t+1:n})^{-1} \mathbf{X}_{1:n}^{\mathrm{T}} \mathbf{X}_{1:n} (\mathbf{X}_{1:t+1}^{\mathrm{T}} \mathbf{X}_{1:t+1})^{-1} \mathbf{x}_{t+1}^{\mathrm{T}}} \,.$$

Using binary search over $W(z)$, it is computationally feasible to derive the z_0-quantile for V given the level z_0. This z_0-quantile will be referred to as the *Worsley's z_0-critical value*.

If we need to work with the test (5) instead of (4), i.e. if we are restricted to a subset of observations, the Worsley's approximation gets the form

$$W_{i:j}(z) = \Pr[V_{i:j} \le z] \approx B_{1,\,(j-i+1)/2-2}(z)$$
$$-\frac{2\beta_{3/2,\,(j-i+1)/2-1}}{\pi(j-i-1)} \times \left(\sum_{t=i+1}^{j-3} \xi_t^{i:j} - \frac{1}{6} \left(\frac{j-i-4}{6} \times \frac{z}{1-z} - 1 \right) \times \sum_{t=i+1}^{j-3} (\xi_t^{i:j})^3 \right),$$

where

$$\xi_t^{i:j} = \sqrt{\mathbf{x}_{t+1}^{\mathrm{T}} (\mathbf{X}_{t+1:j}^{\mathrm{T}} \mathbf{X}_{t+1:j})^{-1} \mathbf{X}_{i:j}^{\mathrm{T}} \mathbf{X}_{i:j} (\mathbf{X}_{i:t+1}^{\mathrm{T}} \mathbf{X}_{i:t+1})^{-1} \mathbf{x}_{t+1}^{\mathrm{T}}} \,.$$

The z_0-quantile derived by binary search over $W_{i:j}(z)$ will be denoted as $W_{i:j}^{-1}(z)$.

If H is rejected, then (4) also suggests a natural estimator of the unknown value τ of the form

$$\hat{\tau} = \underset{t \in \{3, \ldots, n-3\}}{\arg\max}\, V_t \,,$$

or, in the restricted form,

$$\hat{\tau}_{i:j} = \underset{t \in \{i+2, \ldots, j-3\}}{\arg\max}\, V_t^{i:j} \,.$$

4 An iterative approach

Assume that the historical, say daily data $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are sorted in the way that (x_1, y_1) are the most recent (say, today's) prices and (x_n, y_n) are the endmost prices. With this convention we have $T^* = 1$ (today), $T_0 > 1$ (day of hedge inception) and n = the endmost observation available (say, quoted several years ago).

The simplest approach is: test H using the statistic $V_{1:T_0}$ according to (5) at a chosen level z_0; the critical region is given by the condition $V_{1:T_0} > W_{1:T_0}^{-1}$. However, this simple approach suffers from (at least) two drawbacks.

i) If the changepoint τ exists and it occurred in time T_0 or in a short period afterwards, then the test probably will not be able to detect it.
ii) There may have been more than one changepoint in the period $\{1, ..., T_0\}$, but we constructed the test using the single-changepoint alternative.

We overcome the drawbacks in the following way. We apply the test iteratively. We choose a small constant q, say $q = 20$ and run the following algorithm:

1. **for** $t := q$ **to** n **do**
2.   **if** $V_{1:t} > W_{1:t}^{-1}(z_0)$ **then**
3.   **if** $\hat{\tau}_{1:t} \ge T_0$ **then**
4.   **report** "changepoint $\hat{\tau}_{1:t}$ detected"
5.   estimate λ using only the data $\{1, 2, \ldots, \hat{\tau}_{1:t}\}$
6.   **else**
7.   **report** "no changepoint after T_0 detected"
8.   **end if**
9.   **stop**
10.   **end if**
11. **next** t
12. **report** "no changepoint found" and **stop**.

The procedure processes the data iteratively. First, only a recent data history is taken into account; then, as t grows, the data history involved is longer. When we first exceed the critical value, we declare that the point of change has been detected. Observe that if (ii) holds, this procedure is likely to detect *the most recent* changepoint. This point is crucial for correctness of the step 5.

If the estimated point of change occurred before T_0, then the situation is "safe" (step 7) but if the estimated point of change occurred at T_0 or later, the step 5 estimates the new value of hedge ratio λ (say, using (3)) using only the data after the point of change. Then the risk manager assesses whether the newly estimated value of λ differs significantly from its original value determined at the hedge inception. Then she/he can adjust the volume of the hedging derivative appropriately.

5 Example

To visualize the method we plot the processes

$$\tilde{V}_t = t \times V_{1:t}, \quad \tilde{W}_t = t \times W_{1:t}^{-1}(z_0), \quad \tilde{\tau}_t = \hat{\tau}_{1:t}$$

for $t = q, q+1, \ldots$, where $q = 20$, with the choices $z_0 = 5\%$-*level* and $z_0 = 1\%$-*level*. (The scaling factor t in the definition of $\tilde{V}_t$ and $\tilde{W}_t$ has been added, without loss of generality, to make Figure 2 more transparent.)

Such a plot also shows how stable the value $\hat{\tau}_{1:t}$ output in step 4 is. (We say that the value $\hat{\tau}_{1:t}$ is *fully stable* if, for any t and t' for which the conditions $V_{1:t} > W_{1:t}^{-1}(z_0)$ and $V_{1:t'} > W_{1:t'}^{-1}(z_0)$ are met, it holds $\hat{\tau}_{1:t} = \hat{\tau}_{1:t'}$.) Of course, by the stochastic nature of data, we cannot expect full stability; but we roughly say that the estimator $\hat{\tau}_{1:t}$ is stable if it does not significantly change even when a longer history of data is taken into account in the test.

Results of a simulated example are shown in Figure 3. We generated a trajectory of kerosene x_t for $t = 1, \ldots, 500$ (which corresponds to two years if 1 year = 250 business days) as a lognormal random walk varying between \$20 and \$85, see Figure 2. The crude oil process y_t was simulated using (3) with $\nu_t \sim N(0, 1)$ independent and

$$\sigma = 0.1, \alpha = 0, \lambda = \begin{cases} 1.4 & \text{for } t \in \{1, \ldots, 169\}, \\ 1.3 & \text{for } t \in \{170, \ldots, 500\}. \end{cases} \quad (6)$$

Observe that the variance is quite high: if the price of x is \$100, the standard error is \$10.

Assume that the time of inception of the hedge was $T_0 = 250$ (that is, the risk-management strategy was commenced one year ago). At that time, say that the hedge ratio was chosen at the level $\lambda_{original} = 1.2$.

In Figure 3 it can be seen that the procedure detects $\hat{\tau}_{1:t} = 134$. Hence we have an indicator that over the last year, a change has indeed occurred. Observe that 134 is an inexact estimate – by (6) we know that the point of change appeared 169 days ago.

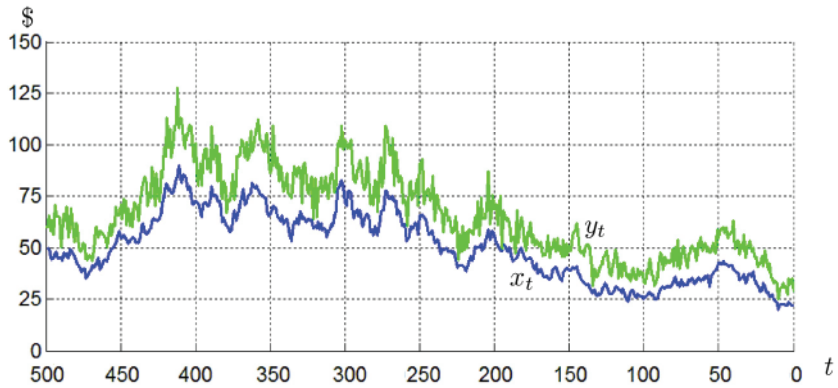
If we don't stop in step 9 when the 1% level is first exceeded and iterate further, we arrive at the estimate $\hat{\tau}_{1:t} = 149$ (which is a value closer to the true value 169).

It can be seen in Figure 3 that the estimate $\hat{\tau}_{1:t} = 149$ is stable. Hence, the method suggests to re-estimate the hedge ratio in step 5 either using either last 133 or last 148 observations, the remaining ("old" ones, i.e. the observations before the estimated point of change) being omitted.

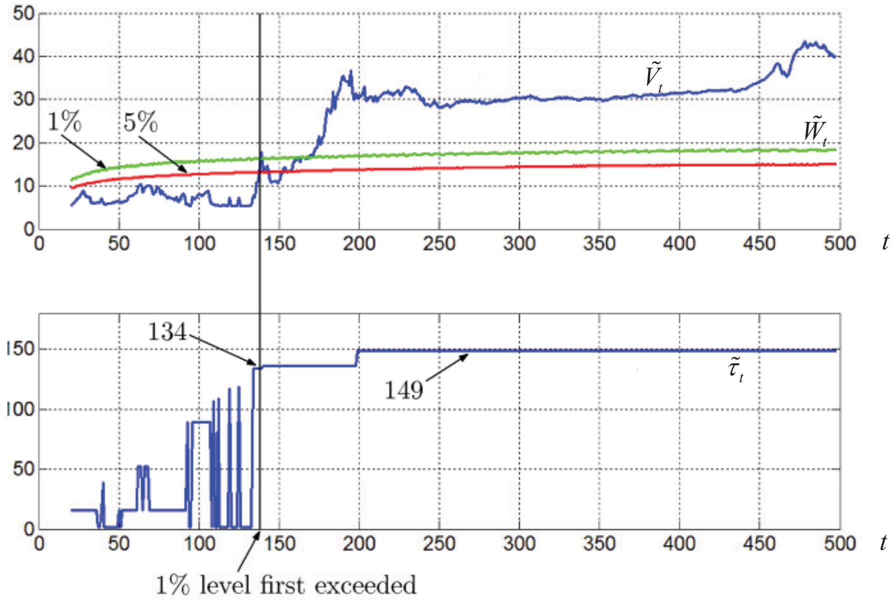
We know that the true value of the contemporary hedge ratio is 1.4. The resulting OLS-estimates based on (3) are

$$\hat{\lambda}_{\text{from data } t \in \{1, \ldots, 133\}} = 1.43 \text{ and } \hat{\lambda}_{\text{from data } t \in \{1, \ldots, 148\}} = 1.46.$$

Hence the risk manager is now suggested to adjust the volume of the hedging derivative from the original volume $\lambda_{original} = 1.2$t or crude oil per ton of kerosene to $\lambda_{new} = 1.43$t or 1.46t of crude oil per ton of kerosene.



2: *Simulated evolution of prices of kerosene (y_t) and the crude oil (x_t) as a function of time (t)*



3: *The process $\tilde{V}_t$, Worsley's 5% and 1% critical values ($\tilde{W}_t$) and the process $\tilde{\tau}_t$*

6 SUMMARY AND CONCLUSIONS

We have proposed an iterative approach for detecting a possible point of change in the hedge ratio. The approach helps in detection whether the hedge relationship should be adjusted before maturity to achieve better hedge effectiveness. Several questions remain open. In particular, the estimator $\hat{\tau}_{1:t}$ of the point of change is not exact. Therefore, when estimating the adjusted value of the hedge ratio, it might be also appropriate to take into account the possible error in $\hat{\tau}_{1:t}$. Another problem is that the test, by its essence, is likely to detect the existence of the point of change after some time of its occurrence. (Fortunately, the more significantly the hedge ratio changes, the earlier the test will detect the change.) It would be suitable to analyze the delay in more detail and modify the method in a way making the delay as short as possible. It would be also suitable to quantify possible losses induced both by the imprecise estimate $\hat{\tau}_{1:t}$ and by the delayed detection of the point of change.

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