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PARAMETERIZATION OF ORTHONORMAL THIRD-ORDER MATRICES FOR LINEAR CALIBRATION

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Abstract

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The paper derives a parametric definition of the set of third-order orthonormal real matrices.
The derivation is done in several partial steps. First a generalized unit matrix is introduced as the simplest case of an orthonormal matrix along with some of its properties and, subsequently, the properties of orthonormal matrices are proved that will be needed.
The derivation itself of a parametric definition of third-order orthonormal matrices is based on the numbers of zero entries that are theoretically possible. Therefore, it is first proved that a third-order square matrix with the number of non-zero entries different from nine, eight, five, or three cannot be orthonormal.
The number of different ways in which the set of third-order orthonormal matrices can be parameterized is greater than one. The concepts of a rotation matrix and a flop-enabling rotation matrix are introduced to motivate the parameterization chosen.
Given the product of two rotation matrices and one flop-enabling rotation matrix, it is first proved that it is a third-order orthonormal matrix. In the last part of the paper, it is then proved that such a product already includes, as special cases, all the third-order orthonormal matrices. It is thus a parametric definition of all third-order orthonormal matrices.

orthonormal matrix, parameterization, rotation matrix, rotation matrix with possible axis polarity change

Calibration approach is often used in processing data obtained from multiple sources or by multiple different procedures. It may be encountered in a number of disciplines ranging from engineering to medicine where it is used for diagnostic purposes.

The most precisely frequently used calibration models are linear ones. What all the calibration models have in common is that they are designed to find (use input data to estimate) a real matrix which is then used as a basis of what is called a calibration function (Myšková, 2007, 2006).

If the model of a real-life situation can be interpreted in such a way that none of the data acquiring procedures chosen systematically distorts the real-life situation (or all the data acquiring procedures chosen do distort the real-life situation basically in the same way) and no other a priori information is known about the real-life situation which could be added as additional conditions of the model, then we say that this is a linear calibration of non-specified identical objects (Moll, Myšková; 2007). The fact that such objects are identical formulated mathematically then means that the calibration matrix is orthonormal.

Estimating the parameters of a linear calibration model always requires solving a nonlinear minimization problem. Generally, such problems tend to be rather sensitive to more parameters being in-

troduced than necessary. Minimization problems with more parameters and additional conditions are much more complex than those with less parameters and no additional conditions ([Moll, Myšková; 2007). This is the reason why the set of orthonormal matrices should be parameterized with a minimum of parameters and with no additional conditions. The set of two-dimensional orthonormal matrices with identical determinant can be described using a single parameter and, among all the different parameterizations of this set, usually it is no problem to find a bijective mapping. However, three-dimensional orthonormal matrices already lack these properties favourable for parameterization.

MATERIAL AND METHODS – BASIC CONCEPTS

Definition 1: A square (n/n) real matrix M is orthonormal if $MM^T = M^TM = E$ where E is a unit (n/n) matrix.

Note: From Definition 1 immediately follows that $\det i \in \{-1, 1\}$.

Definition 2: An (n/n) matrix with exactly n non-zero entries from the set $\{-1, 1\}$ such that no two non-zero entries are on a single row and no two non-zero entries are in a single column will be called a generalized unit matrix.

Theorem 3: Let M be an orthonormal (n/n) matrix with entries m_{ij}. Then:

a. All the entries of M are in the interval $[-1, 1]$.
b. Let, for some $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$, $|m_{ij}| = 1$. Then all the other entries of M in row i and column j are zeros.
c. Let $n \geq 2$ and let exists with an entry m_{ij} for which $|m_{ij}| = 1$. Let matrix N be created by striking out row i and column j from matrix M. Then N is orthonormal.
d. Let Mbe different from generalized unit matrix. Create matrix N by striking out all rows and all columns from matrix M containing an entry from the set $\{-1, 1\}$. Then N is orthonormal.

Proof:

Ad a) Suppose that, for a fixed entry m_{ij} of matrix M, we have $|m_{ij}| = 1$.Then, in matrix MM^T, the entry at the i-th diagonal position is greater than one, which contradictory to the definition of an orthonormal matrix M.

Ad b) Let one of the other entries of matrix Mon row i or column j, say m_{ij}, be non-zero. Then $MM^T = E$ is not true.

Ad c) According to what was said the above, we have $M = \begin{pmatrix} M_{11} & \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} & M_{12} \\ (0\,..\,0) & m_{ij} & (0\,..\,0) \\ M_{21} & \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} & M_{22} \end{pmatrix}$. Hence

$$E = MM^T = \begin{pmatrix} M_{11}M_{11}^T + M_{12}M_{12}^T & \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} & M_{11}M_{21}^T + M_{12}M_{22}^T \\ (0\,..\,0) & 1 & (0\,..\,0) \\ M_{21}M_{11}^T + M_{22}M_{12}^T & \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} & M_{21}M_{21}^T + M_{212}M_{22}^T \end{pmatrix}.$$

The proof follows immediately from the equation $NN^T = \begin{pmatrix} M_{11}M_{11}^T + M_{12}M_{12}^T & M_{11}M_{21}^T + M_{12}M_{22}^T \\ M_{21}M_{11}^T + M_{22}M_{12}^T & M_{21}M_{21}^T + M_{212}M_{22}^T \end{pmatrix}$ and from the previous equation.

Ad d) The proof follows from the previous item.

Theorem 4: Let M and N be two orthonormal matrices of the same size. Then MN is an orthonormal matrix.

Proof: The theorem is well known and its proof is very simple: $MN(MN)^T = MNN^TM^T = MEM^T = MM^T = E$.

RESULTS – DERIVING A PARAMETERIZATION

Theorem 5: Let M be a (3/3) orthonormal matrix. Then M has at most six zero entries.

Proof: Suppose that a (3/3) matrix M has more than six zero entries. It is easy to see that $\det(M) = 0$. This is, however, in contradiction to the properties of M as an orthonormal matrix.

Theorem 6: Let M be a (3/3) orthonormal matrix having exactly six zero entries. Then three of its non-zero entries are in the set {–1, 1} and no two non-zero entries are on the same row and no two non-zero entries are in the same column (this means that $M = E^*$).

Proof: Let at least two from three of the non-zero entries of M be on the same row (in the same column). It is easy to see that then det(M) = 0and so M is not orthonormal. The fact that three of the non-zero entries are in the set {–1, 1} follows immediately from the equation $MM^T = E$.

Theorem 7: A (3/3) orthonormal matrix M cannot have exactly two entries from the set {–1, 1}.

Proof: Let M has exactly two entries in the set {–1, 1}. Then, by Theorem 3b), these entries are neither on the same row nor in the same column. By striking out the rows and columns containing such entries, a (1/1) orthonormal matrix is obtained. There are, however, exactly two such matrices – namely (1) and (–1). Thus, M contains a third entry from the set {–1, 1}.

Theorem 8: Let M be a (3/3) orthonormal matrix. Then the number of its zero entries is in the set {0, 1, 4, 6}.

Proof:

a. The matrix $M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{5}{8} & \frac{3\sqrt{3}}{8} \\ \frac{3}{4} & -\frac{3\sqrt{3}}{8} & -\frac{1}{8} \end{pmatrix}$ has no zero entry. By direct calculation it can be verified that it is orthonormal.

b. The matrix $M = \begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \end{pmatrix}$ has one zero entry. By direct calculation it can be verified that it is orthonormal.

c. Let a matrix M have exactly two zero entries. These zero entries are neither on the same row nor in the same column. If they were on the same row (in the same column), then the only non-zero entry on this row (in this column) would be an element u of the set {–1, 1}. By Theorem 3b) then, M has at least four zero entries.

Let the zero entries of M be m_{ij}, m_{kl} for $i \neq k, j \neq l, i, j, k, l \in \{1, 2, 3\}$. Let $r \in \{1, 2, 3\} - \{j, l\}$. Then, since M is orthonormal, we have $m_{ir} \cdot m_{kr} = 0$,which is in contradiction to the assumption. Thus M cannot have exactly two non-zero entries.

d. Let the M have exactly three zero entries. Then these zero entries must be on different rows and in different columns, which can be proved in much the same way as above and again, as above, it can be shown that such a situation cannot occur.

e. The matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ has four zero entries. By direct calculation it can be verified that it is orthonormal.

f. Let a matrix M have exactly five zero entries. Then two columns contain two zero entries each and the absolute value of the remaining entry in these columns equals one. This means that M has two more non-zero entries. By Theorem 3b), at least one of these must be equal one and thus M must have at least six zero entries, which is a contradiction.

g. The unit (3/3) matrix is orthonormal and has six zero entries.

h. By Theorem 5, a matrix M cannot have more than six zero entries.

Definition 9: Let $n \geq 2$ be a natural number.

a. We denote by $X^n_{p,q}(\alpha)$ an (n/n) matrix $(x_{ij})^n_{i,j=1}$ in which, for $\{p, q\} \in \{1, 2, \ldots, n\}^2$, $p < q$; $x_{ii} = 1$ for $i \in \{1, 2, \ldots, n\} - \{p, q\}$; $x_{pp} = x_{qq} = \cos\alpha$, $x_{pq} = \sin\alpha$, $x_{qp} = -\sin\alpha$; and $x_{ii} = 0$ for the remaining index pairs, i.e. for $(i, j) \in \{1, 2, \ldots, n\}^2 - \{(1, 1), (2, 2), \ldots, (n, n), (p, q), (q, p)\}$. We will call $X^n_{p,q}(\alpha)$ a matrix of α-rotation between axis q and axis p or rotation matrix for short.

b. Let $X^n_{p,q}(\alpha)$ be a matrix of α-rotation between axis q and axis p and $r \in \{1, 2, \ldots, n\} - \{p, q\}$. If, in $X^n_{p,q}(\alpha)$, the entry on row r and in column r for one chosen r is replaced by the symbol ε_r, the resulting matrix will be called an r-flop-enabling matrix of α-rotation between axis q and axis p and denoted $X^n_{p,q}(\alpha; \varepsilon_r)$.

Note: Putting $\varepsilon_r = -1$ will cause a change in the polarity of axis r. Putting $\varepsilon_r = 1$ means that the polarity of axis r remains the same.

Example: Let $n = 3$. For a given fixed α, there are three matrices of α-rotation. Next we will only use the matrices $X^3_{1,2}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $X^3_{2,3}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$.

Note: Theorems 10 to 13 can be proved by direct computation.

Theorem 10: Each rotation matrix and each flop-enabling rotational matrix in which either $\varepsilon_r = 1$ or $\varepsilon_r = -1$ is orthonormal.

Theorem 11: Let, for a natural number $n \geq 2$, $X^n_{p,q}(\alpha)$ be a matrix of α-rotation between axis q and axis p and $X^n_{p,q}(\beta)$ a matrix of β-rotation between axis q and axis p. Then both matrix $X^n_{p,q}(\alpha) \cdot X^n_{p,q}(\beta)$ and matrix $X^n_{p,q}(\beta) \cdot X^n_{p,q}(\alpha)$ are matrices of $(\alpha + \beta)$-rotation between axis q and axis p, that is, $X^n_{p,q}(\alpha) \cdot X^n_{p,q}(\beta) = X^n_{p,q}(\alpha + \beta)$ and $X^n_{p,q}(\beta) \cdot X^n_{p,q}(\alpha) = X^n_{p,q}(\alpha + \beta)$.

Corollary 12: Let, for a natural number $n \geq 2$, $X^n_{p,q}(\alpha)$ be a rotation matrix. Then the matrix $X^n_{p,q}(-\alpha)$ is a matrix inverse to $X^n_{p,q}(\alpha)$.

Theorem 13: Let $n = 3$, $X^3_{2,3}(\vartheta)$ be a matrix of ϑ-rotation between the third and second axes, $X^3_{1,2}(\varphi; \varepsilon_3)$ be a matrix of φ-rotation between the second and first axes with a change in the polarity of the third axis if necessary, and $X^3_{2,3}(\alpha)$ be a matrix of α-rotation between the third and second axes, then $X^3_{2,3}(\vartheta) \cdot X^3_{1,2}(\varphi; \varepsilon_3) \cdot X^3_{2,3}(\alpha)$ has the form $\begin{pmatrix} \cos\varphi & \cos\alpha\cdot\sin\varphi & \sin\alpha\cdot\sin\varphi \\ -\cos\vartheta\cdot\sin\varphi & \cos\vartheta\cdot\cos\alpha\cdot\cos\varphi - \varepsilon_3\sin\vartheta\cdot\sin\alpha & \cos\vartheta\cdot\sin\alpha\cdot\cos\varphi + \varepsilon_3\sin\vartheta\cdot\cos\alpha \\ \sin\vartheta\cdot\sin\varphi & -\sin\vartheta\cdot\cos\alpha\cdot\cos\varphi - \varepsilon_3\cos\vartheta\cdot\sin\alpha & -\sin\vartheta\cdot\sin\alpha\cdot\cos\varphi - \varepsilon_3\cos\vartheta\cdot\cos\alpha \end{pmatrix}$.

Note: The above theorem is shown as a motivation to the selection of parameterization in the following theorem.

Theorem 14: Let M be a (3/3) orthonormal matrix with all its entries are non-zero or containing a single zero entry which is one of the entries m_{22}, m_{23}, m_{32}, m_{33} and $\varepsilon \in \{-1, 1\}$. Then there exists a triple $(\varphi, \vartheta, \alpha) \in \left\{(0, \pi) - \left\{\frac{\pi}{2}\right\}\right\} \times \left\{(0, 2\pi) - \left\{\frac{\pi}{2}, \pi\frac{3\pi}{2}\right\}\right\} \times \left\{(0, 2\pi) - \left\{\frac{\pi}{2}, \pi\frac{3\pi}{2}\right\}\right\}$ such that

$$M = \begin{pmatrix} \cos\varphi & \cos\alpha\cdot\sin\varphi & \sin\alpha\cdot\sin\varphi \\ -\cos\vartheta\cdot\sin\varphi & \cos\vartheta\cdot\cos\alpha\cdot\cos\varphi - \varepsilon\cdot\sin\vartheta\cdot\sin\alpha & \cos\vartheta\cdot\sin\alpha\cdot\cos\varphi + \varepsilon\cdot\sin\vartheta\cdot\cos\alpha \\ \sin\vartheta\cdot\sin\varphi & -\sin\vartheta\cdot\cos\alpha\cdot\cos\varphi - \varepsilon\cdot\cos\vartheta\cdot\sin\alpha & -\sin\vartheta\cdot\sin\alpha\cdot\cos\varphi - \varepsilon\cdot\cos\vartheta\cdot\cos\alpha \end{pmatrix}.$$

Proof: Let M be an arbitrary real orthonormal (3/3) matrix, $M = (m_{ij})_{i,j=1}^3$ with all its entries being non-zero or only with the entry m_{33} being zero. Then each of the M entries is in the interval (–1/1). The function $f(x) = \cos x$ is an injection in $(0, \pi)$ mapping it onto the interval (–1/1). Therefore, without loss of generality, an arbitrary but fixed $\varphi \in (0, \pi)$ can be chosen assuming that $m_{11} = \cos\varphi$. Since $m_{11} \neq 0$, we have $\varphi \neq \frac{\pi}{2}$.

Further, $m^2_{11} + m^2_{12} + m^2_{13} = 1$ implies $m^2_{21} + m^2_{31} = 1 - m^2_{11} = 1 - \cos^2\varphi = \sin^2\varphi$. Hence $m^2_{21} = \sin^2\varphi - m^2_{31}$. The last equation implies $|m_{21}| \leq |\sin\varphi|$. Again, using the fact that $f(x) = \cos x$ is an injection in $[0, \pi]$ mapping it onto the interval $[-1, 1]$ and choosing an arbitrary but fixed $\vartheta \in (0, \pi)$ without loss of generality, we can assume that $m_{21} = -\sin\varphi\cdot\cos\vartheta$. The condition $m_{21} \neq 0$ implies $\vartheta \neq \frac{\pi}{2}$. Again, we can use the equation $m^2_{11} + m^2_{12} + m^2_{13} = 1$ obtaining $m^2_{31} = 1 - m^2_{11} = 1 - \cos^2\varphi - \sin^2\varphi\cdot\cos^2\vartheta = \sin^2\varphi\cdot\sin^2\vartheta$. Hence $|m_{31}| \leq |\sin\varphi\cdot\sin\vartheta|$. The last equation can be satisfied in two different ways. Either $m_{31} \leq \sin\varphi\cdot\sin\vartheta$ or $m_{31} \leq -\sin\varphi\cdot\sin\vartheta$. These two equations can be written as $m_{31} \leq \sin\varphi\cdot\sin\vartheta$ provided that the set of the possible values of ϑ is extended to $\vartheta \in (0, 2\pi) - \left\{\frac{\pi}{2}, \pi\frac{3\pi}{2}\right\}$.

Now reasoning similar to that used for column one can also be applied to row one of the matrix M. This yields $m_{12} \leq \sin\varphi\cos\alpha$ and $m_{13} \leq \sin\varphi\sin\alpha$ for $\alpha \in (0, 2\pi) - \left\{\frac{\pi}{2}, \pi\frac{3\pi}{2}\right\}$.

Denoting $m_{22} = a$, we can use the equation $m^2_{12} + m^2_{22} + m^2_{32} = 1$ to obtain $m^2_{32} = 1 - m^2_{12} - m^2_{22} = 1 - \sin^2\varphi\cos^2\alpha - a^2$. Hence either $m_{32} = \sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2}$ or $m_{32} = -\sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2}$. Thus $m_{32} = \varepsilon_3\sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2}$

where $\varepsilon_3 \in \{-1, 1\}$. By the same reasoning, we can use the equation $m_{21}^2 + m_{22}^2 + m_{23}^2 = 1$ to obtain $m_{23} = \varepsilon_4\sqrt{1 - \sin^2\varphi.\cos^2\vartheta - a^2}$ where $\varepsilon_4 \in \{-1, 1\}$.

Let us now summarize the preceding reasoning in a matrix M. Denoting $m_{33} = b$ we can write $M = \begin{pmatrix} \cos\varphi & \sin\varphi\cdot\cos\alpha & \sin\varphi\cdot\sin\alpha \\ -\sin\varphi\cdot\cos\vartheta & a & \varepsilon_4\sqrt{1 - \sin^2\varphi.\cos^2\vartheta - a^2} \\ \sin\varphi\cdot\sin\vartheta & \varepsilon_3\sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2} & b \end{pmatrix}$.

The scalar product of the first two columns of M is zero. This leads to an equation $\sin\varphi\cdot\cos\varphi\cdot\cos\alpha - a\cdot\sin\varphi\cdot\cos\vartheta + \varepsilon_3\sin\varphi\cdot\sin\vartheta\sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2} = 0$ or to an equation $\sin\varphi\cdot\cos\varphi\cdot\cos\alpha - a\cdot\sin\varphi\cdot\cos\vartheta = -\varepsilon_3\sin\varphi\cdot\sin\vartheta\sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2}$. Squaring both sides of the equation yields $\sin^2\varphi\cdot\cos^2\varphi\cdot\cos^2\alpha + a^2\sin^2\varphi\cdot\cos^2\vartheta - 2a\sin^2\varphi\cdot\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta = \sin^2\varphi\cdot\sin^2\vartheta\cdot(1 - \sin^2\varphi\cdot\cos^2\alpha - a^2)$.

Since $\sin\varphi \neq 0$, the last equation can be reduced by $\sin^2\varphi$, which yields $\cos^2\varphi\cdot\cos^2\alpha + a^2\cdot\cos^2\vartheta - 2a\cdot\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta = \sin^2\vartheta\cdot(1 - \sin^2\varphi\cdot\cos^2\alpha - a^2)$. Hence $\cos^2\varphi\cdot\cos^2\alpha + a^2\cdot\cos^2\vartheta - 2a\cdot\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \sin^2\vartheta + \sin^2\vartheta\cdot\sin^2\varphi\cdot\cos^2\alpha + a^2\cdot\sin^2\vartheta = 0$. After some simplification then $a^2 - 2a\cdot\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta + (\cos^2\varphi\cdot\cos^2\alpha - \sin^2\vartheta + \sin^2\vartheta\cdot\sin^2\varphi\cdot\cos^2\alpha) = 0$.

Now let us view the last equation as a quadratic equation with the unknown a in the form $a^2 - 2aK + L = 0$. Its solutions can be written as $a = K \pm \sqrt{K^2 - L}$. We have $\sqrt{K^2 - L} = \sqrt{\cos^2\varphi.\cos^2\alpha.\cos^2\vartheta - (\cos^2\varphi\cdot\cos^2\alpha - \sin^2\vartheta + \sin^2\vartheta\cdot\sin^2\varphi\cdot\cos^2\alpha)}$, hence $\sqrt{K^2 - L} = \sqrt{-\cos^2\varphi.\cos^2\alpha.\sin^2\vartheta + \sin^2\vartheta - \sin^2\vartheta\cdot\sin^2\varphi\cdot\cos^2\alpha)}$, which then yields $\sqrt{K^2 - L} = |\sin\alpha\cdot\sin\vartheta|$. Thus, we have $a = \cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta$ where $\varepsilon \in \{-1, 1\}$. Hence $m_{32} = \varepsilon_3\sqrt{1 - \sin^2\varphi.\cos^2\alpha - a^2} = \varepsilon_3\sqrt{1 - \sin^2\varphi.\cos^2\alpha - (\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta)^2}$ and $m_{32} = \varepsilon_3\sqrt{(\varepsilon\sin\alpha\cdot\cos\vartheta + \cos\varphi\cdot\cos\alpha\cdot\sin\vartheta)^2}$, which means that $m_{32} = \pm(\varepsilon\sin\alpha\cdot\cos\vartheta + \cos\varphi\cdot\cos\alpha\cdot\sin\vartheta)$.

Substituting this result into the matrix again, we get

$$M = \begin{pmatrix} \cos\varphi & \sin\varphi\cdot\cos\alpha & \sin\varphi\cdot\sin\alpha \\ -\sin\varphi\cdot\cos\vartheta & \cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta & \varepsilon_4\sqrt{1 - \sin^2\varphi.\cos^2\vartheta - a^2} \\ \sin\varphi\cdot\sin\vartheta & \pm(\varepsilon\cdot\sin\alpha\cdot\cos\vartheta + \cos\varphi\cdot\cos\alpha\cdot\sin\vartheta) & b \end{pmatrix}.$$

The scalar product of the first two columns of M is zero. This will give us the sign of m_{23} with $m_{32} = -\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\cdot\sin\alpha\cdot\sin\vartheta$.

After substitution, matrix M will have the form

$$\begin{pmatrix} \cos\varphi & \sin\varphi\cdot\cos\alpha & \sin\varphi\cdot\sin\alpha \\ -\sin\varphi\cdot\cos\vartheta & \cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta & \varepsilon_4\sqrt{1 - \sin^2\varphi.\cos^2\vartheta - (\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta)^2} \\ \sin\varphi\cdot\sin\vartheta & -\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta & b \end{pmatrix}.$$ Since $m_{23} = \varepsilon_4\sqrt{1 - \sin^2\varphi.\cos^2\vartheta - (\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta)^2}$ we can write $m_{23} = \varepsilon_4\sqrt{(\varepsilon\cos\alpha\cdot\sin\vartheta + \cos\varphi\cdot\sin\alpha\cdot\cos\vartheta)^2}$ and so $m_{23} = \pm(\varepsilon\cos\alpha\cdot\sin\vartheta + \cos\varphi\cdot\sin\alpha\cdot\sin\vartheta)$.

After substitution, matrix M will have the form

$$\begin{pmatrix} \cos\varphi & \sin\varphi\cdot\cos\alpha & \sin\varphi\cdot\sin\alpha \\ -\sin\varphi\cdot\cos\vartheta & \cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta & \pm(\cos\varphi\cdot\sin\alpha\cdot\cos\vartheta + \varepsilon\cos\alpha\cdot\sin\vartheta) \\ \sin\varphi\cdot\sin\vartheta & -\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\sin\alpha\cdot\sin\vartheta & b \end{pmatrix}.$$ We have

$b^2 = 1 - \sin^2\varphi\sin^2\alpha - (\cos\varphi\cdot\sin\alpha\cdot\cos\vartheta + \varepsilon\cos\alpha\cdot\sin\vartheta)^2$, $b^2 = (\varepsilon\cos\alpha\cdot\cos\vartheta - \cos\varphi\cdot\sin\alpha\cdot\sin\vartheta)^2$ and hence $b = \pm(\cos\varphi\cdot\sin\alpha\cdot\sin\vartheta - \varepsilon\cos\alpha\cdot\cos\vartheta)$.

This means that $M = \begin{pmatrix} \cos\varphi & \sin\varphi\cdot\cos\alpha & \sin\varphi\cdot\sin\alpha \\ -\sin\varphi\cdot\cos\vartheta & \cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\cdot\sin\alpha\cdot\sin\vartheta & \pm(\cos\varphi\cdot\sin\alpha\cdot\cos\vartheta + \varepsilon\cdot\sin\vartheta\cdot\cos\alpha) \\ \sin\varphi\cdot\sin\vartheta & -\cos\varphi\cdot\cos\alpha\cdot\cos\vartheta - \varepsilon\cdot\sin\alpha\cdot\sin\vartheta & \pm(\cos\varphi\cdot\sin\alpha\cdot\sin\vartheta + \varepsilon\cdot\cos\alpha\cdot\cos\vartheta) \end{pmatrix}$.

The equation

$$\cos\varphi\cdot\sin\varphi\cdot\sin\vartheta - \sin\varphi\cdot\cos\vartheta[\pm(\cos\varphi\cdot\sin\alpha\cdot\cos\vartheta + \varepsilon\cos\alpha\cdot\sin\vartheta)] + \sin\alpha\cdot\sin\vartheta[\pm(\cos\varphi\cdot\sin\alpha\cdot\sin\vartheta - \varepsilon\cos\alpha\cdot\cos\vartheta)] = 0$$

yields immediately

$m_{23} = \cos\vartheta\cdot\sin\alpha\cdot\cos\varphi + \varepsilon\cdot\sin\vartheta\cdot\cos\alpha$, $m_{33} = -\sin\vartheta\cdot\sin\alpha\cdot\cos\varphi + \varepsilon\cdot\cos\vartheta\cdot\cos\alpha$.

The theorem is proved for the case of all the entries of M being non-zero or only entry m_{33}being zero. If some of the entries m_{23}, m_{32}, m_{22} is zero, we can proceed in much the same way.

Note: Let M is the same matrix as in Theorem 14, then $\det M = \varepsilon$.

Note: The assumption that only one of the entries m_{22}, m_{23}, m_{32}, m_{33} is zero is not substantial and the theorem may be extended without change to all (3/3) orthonormal matrices with one zero entry. Using a special choice of parameters, also all (3/3) orthonormal matrices that contain more than one zero entry can be written in the form shown above. This parameter choice, however, is not unique in all cases.

SUMMARY

The paper deals with the need to parameterize three-dimensional orthonormal matrices in terms of linear calibration. The set of orthonormal matrices is parameterized with three continuous and one discrete parameter. The meaning of the continuous parameters is obvious from the introduced concept of a rotation matrix while the meaning of the discrete parameter is explained by the rotation matrix with a change of axis polarity if necessary.

SOUHRN

Parametrizace ortonormálních matic třetího řádu pro lineární kalibraci

Práce řeší potřebu parametrizace trojrozměrných ortonormálních matic z hlediska lineární kalibrace. Množina ortonormálních matic je parametrizována třemi spojitými a jedním diskrétním parametrem. Smysl jednotlivých spojitých parametrů je zřejmý ze zavedeného pojmu matice otočení a význam diskrétního parametru vysvětluje matice otočení s případnou změnou polarity osy.

ortonormální matice, parametrizace, matice otočení, matice otočení s případnou změnou polarity osy

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