FUNCTIONS OF SEVERAL VARIABLES ANALYSIS APPLIED IN INVENTORY MANAGEMENT

Martina Janková, Veronika Novotná, Tereza Varyšová

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Abstract

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In many cases a retailer is not capable of settling an invoice immediately upon receiving it and is given an option by the supplier to settle the invoice within a definite period. The retailer can sell the goods before the deadline, accumulate revenue and earn interest. If the retailer is not able to meet his obligations within the deadline, he is charged an interest. This paper introduces a newly constructed model which enables a retailer to set an optimal price of goods under permissible delay in payments, and to determine the maximum term of payment. The model is based on the assumption of time-dependent demand and has been developed for non-deteriorating goods. The paper further analyzes a situation in which the retailer sell all the goods in time, and a situation in which the deadline was not met. Theoretical results are demonstrated by an illustrative example.

The authors of the paper used methods of analysis and synthesis, and the method of mathematical analysis (differential calculus of multivariable functions, solution of ordinary differential equations). The model suggested in the paper can be expanded in the future. One option is generalization of the model, allowing for the lack of goods, bulk discounts, etc.

inventory management, EOQ model, non-deteriorating goods, local extremes, multivariable functions

Traditional inventory models assume that a retailer pays for the goods the moment they come into inventory. Nowadays, however, it is becoming a common practice that a supplier offers a retailer the option to pay for the goods with a certain delay. The retailer may, before the end of the period, sell the goods, accumulate revenue and earn interest. If the retailer is not able to meet his obligations by the end of the credit period, he is charged an interest. In other words – a supplier provides a retailer with an interest-free credit for a period set in the contract. This paper introduces a newly designed model which enables a retailer to set an optimal price of goods under permissible delay in payments, and to determine the maximum term of payment.

Aim of the paper

The paper's aim is to set up a mathematical model which enables a retailer to fix (based on the knowledge of certain parameters) the optimum selling price per an item of goods and to determine the maximum interval during which goods can be sold at a profit. The model is based on the assumption of time-dependent demand and is developed for non-deteriorating goods. Further assumption is that the inventory is depleted only by demand.

The scientific aim is to verify if such an optimizing problem can be solved. The authors of the paper used analytic and synthetic methods and the method of mathematical analysis (differential calculus of multivariable functions, solution of ordinary differential equations).

The new model is demonstrated in the paper by a specific problem as well as graphical solution.

Research methods

Objective of this paper is to set up a mathematical model which enables a retailer to fix (based on the knowledge of certain parameters) the optimum
selling price per an item of goods and to determine the maximum interval during which goods can be sold at a profit. The model is based on the assumption of time-dependent demand and is developed for non-deteriorating goods. Further assumption is that the inventory is depleted only by demand. The scientific aim is to verify if such an optimizing problem can be solved.

The authors of the paper are using analytic and synthetic methods and the methods of mathematical analysis (differential calculus of multivariable functions, solution of ordinary differential equations). For a deeper analysis the Weierstrass theorem was used: If a function \( f(x) \) is defined and is continuous over a closed bounded interval \([a, b]\), then \( f(X) \) has at least one absolute maximum and minimum point in the interval \([a, b]\).

The authors introduce a model which enables a retailer to set an optimal price of goods under permissible delay in payments and to determine the maximum term of payment. The model is based on the assumption of time-dependent demand and has been developed for non-deteriorating goods.

All calculations were carried out using Maple system, which is used as mathematical software mainly because of its capability to solve problems in a symbolic way. Maple is very similar to programs Mathematica and Maxima, but they offer much less functions. An indisputable advantage of Maple system is that not only it can make analytic calculations with mathematical formulae, but it is also able to make numerical calculus and potential graphical display of results. It is a system with a pleasant user’s environment and, at the same time, a wide range of options for using quantitative methods for practice, application problems, multidisciplinary scientific calculus, etc.

**LITERATURE REVIEW**

It is a natural trend in theoretical basis of scientific disciplines to study various model situations using increasingly precise models which, among other things, enable us to analyse more accurately the simulated processes, to search for more exact meaning of the circumstances under which they run, and to derive practical conclusions, bases, optimum solutions, etc. from the findings. Measurement of economic quantities then corresponds to use of quantitative methods, i.e. methods based on findings of mathematical disciplines. Study of various model situations, focus on simulating conditions, searching for outcomes, optimal solutions and so on are among the most important current trends. For example, McNelis in his paper (2003) applies neural network methodology to inflation forecasting in the Euro-area and the USA. In his paper Zhang (2012), investigated the sensitivity of estimated technical efficiency scores from different methods including stochastic distance function frontier. The paper Ioana et al. (2010) presents a new concept for Fuzzy Logic in economic processes. In related literature Boucekkine et al. (2004) also studied the two-stage optimal control problem involving two deterministic AK models, and Harada (2010) examined the switch from the Solow to AK economies using a similar technique. Many studies have been conducted in order to validate the hysteresis hypothesis, e.g. Alonzo (2011). Stejskal and Stávková (2009) delimitate a partial procedure in the process of modelling and simulation of decision – making results took by individual market subjects.

Inventory management by Nenes et al. (2010) has been recognized as one of the most important functions of industrial and commercial enterprises. Goyal (1985) developed an economic order quantity (EOQ) model under conditions of permissible delay in payments. Dave (1985) corrected Goyal's model by assuming the fact that the selling price is necessarily higher than its purchase price, his viewpoint did not draw much attention to the recent researchers. Jamal et al. (1997) further generalized the model to allow for shortages and deterioration.

Hwang and Shinn (1997) developed the optimal pricing and lot sizing for the retailer under the condition of permissible delay in payments. Liao et al. (2000) developed an inventory model for stock-dependent demand rate when a delay in payment is permissible. All above models (except Dave, 1985) ignored the difference between unit price and unit cost, and obtained the same conclusion as in Goyal (1985). In contrast, Jamal et al. (2000) and Sharker et al. (2000) amended Goyal's model by considering the difference between unit price and cost, and concluded from results that the retailer should settle his account sooner as the unit selling price increases relative to the unit cost. Recently, Teng (2002) provided an alternative conclusion from Goyal (1985), and mathematically proved that it makes economic sense for a well-established buyer to order less quantity and take the benefits of the permissible delay more frequently. Chang et al. (2003) then extended Teng's model and established an EOQ model for deteriorating items.

Polatoglu (1991) examined an inventory model for developing pricing and procurement decisions simultaneously. Under the condition of price-sensitive demand, Weng (1995) considered the supplier's quantity discount from the perspective of reducing the supplier's operating cost and increasing the buyer's demand. Recently, there is a large body of literature on the problem of coordination of replenishment strategies and pricing policies, for example Federgren and Heching (1999), Khouja (2000), Lau and Lau (2003), Maddah and Bish (2004), Banerjee (2005), Chen and Simchi-Levi (2006) and You and Hsieh (2007), etc. However, none of them consider the replenishment policy with a trade credit.

In the classical inventory models the replenishment rate or production rate is often assumed to be constant. However, it has been observed that the production rate is frequently flexible. Silver (1990) discussed the effects of slowing
down production rate in saving potential costs under controllable production rates. Schweitzer and Seidmann (1991) provided the concept of flexibility in production rate and discussed processing rate optimization for a flexible manufacturing system. Bhunia and Maiti (1997) presented two inventory models in which the production rate depends upon the on-hand inventory for the first model and upon the demand for the second one. Manna and Chaudhuri (2001) discussed an EOQ model with deteriorating items in which the production rate is proportional to the time dependent demand rate.

Chu et al. (1998) also examined the economic ordering policy of deteriorating items under the condition of permissible delay in payments. A common assumption of these papers is that the demand is a known constant, which consequently disregards effects of trade credit on the quantity demanded. As implicitly stated by Mehta (1998), a major reason for the supplier to offer a credit period to the retailers is to stimulate the demand for the product he produces. Also, Fewings (1992) stated that the advantage of trade credit for the supplier is substantial in terms of influence on the retailer's purchasing and marketing decisions.

Lately, some researches discussed the impact of delay payment strategy on the inventory models. Abad and Jaggi (2003) provided a seller – buyer inventory model under trade credit. Jaber and Osman (2006) proposed a supplier – retailer supply chain model in which the permissible delay in payments is considered as a decision variable. Inventory by Khaleb (2013) represents one of the most important and difficult assets to be managed at firm level as well as at macro economy level.

**RESULTS AND DISCUSSION**

Assumptions of the model

In this chapter we establish a suitable model for a retailer to determine the optimal price of goods when the supplier offers a permissible delay in payments, and to determine the maximum term of payment. The model has been based on the assumption of time-dependent demand and is developed for non-deteriorating goods. Another assumption is that the inventory is depleted only by demand.

Other assumptions are the following:

- The demand for the item is a downward sloping function of the price and variable time
- Shortage is not allowed
- Time horizon is infinite.

In addition, the following notations are used in the paper:

- **H**: The unit holding cost per year excluding interest charges
- **c**: The unit purchasing cost, with \( c > 0 \)
- **p**: The selling price per unit, \( c < p \)
- **Id**: The interest earned per currency unit per year
- **Ic**: The interest charged per currency unit per stocks per year
- **m**: The period of permissible delay in settling the account, i.e. the trade credit period
- **s**: The ordering costs per order
- **Q**: The order quantity
- **I(t)**: The level of inventory at time \( t (0 \leq t \leq T) \)
- **T**: The replenishment interval \( T > 0 \)
- **D**: The annual demand, as a function of price and time per unit, we set \( D(p, t) = \frac{\alpha p^\beta}{t} \), where \( \alpha > 0 \) and \( \beta > 1 \), \( \alpha \) is a scaling factor, and \( \beta \) is a price-elasticity coefficient. (Ho et al., 2008) Further in the text the notation \( a = \alpha p^\beta \) will be used.

The total annual profit consists of the sales revenue, cost of placing orders, purchasing costs, costs of carrying inventory (excluding interest charges), cost of interest payable for items unsold during the permissible delay (this cost occurs only if \( T > m \)), and the interest earned from sales revenue during the permissible delay period.

**Initial assumptions**

We may assume that the level of inventory \( I(t) \) gradually decreases in time to meet demand. Hence the variation of inventory with respect to time can be determined by the following differential equation:

\[
\frac{dI(t)}{dt} = -D(p,t), 0 \leq t \leq T
\]

\[
\frac{dI(t)}{dt} = -at, 0 \leq t \leq T, a = \alpha p^\beta
\]

The boundary condition is defined as \( I(T) = 0 \).

The retailer sells \( aT/2 \) units of goods per the replenishment cycle time \( T \), which makes sales of \( aT/2 \) per year. The retailer has to pay \( \alpha T/2 \) in full to the supplier by the end of the credit period, which makes \( \alpha T/2 \) in total per year.

As there are many ways to determine the interest payable as well as the interest earned. For simplicity, the authors of the paper used Goyal's method published in (Goyal, 1985).

The total annual profit \( Z(T, p) \) is calculated as

\[
Z(T, p) = \text{Sales revenue} - \text{Costs of placing order} - \text{Costs of purchasing} - \text{Costs of carrying inventory} + \text{interest earned per year}.
\]

**Case in which the goods are sold by the end of the credit period**

If \( T \leq m \), the retailer doesn't incur any additional costs. On the contrary, he may earn interest \( aT/2(m - T/2) \). Consequently, the retailer earns annual interest \( aT/2(m - T/2) \) (Goyal, 1985).
The total annual profit is

\[ Z_1(T,p) = \frac{paT}{2} - \frac{s}{T} - \frac{cT}{2} - \frac{HaT^2}{2} + apI_f \left( m - \frac{T}{2} \right) . \]  

(3)

It follows from (3) that function \( Z_1 \) is a continuous function of two variables, in particular for arbitrary \( T > 0 \) and \( p > 0 \).

In order to model our economic process, it is advisable to limit the value of variables \( T \) and \( p \) in rectangle \( O = \{ (T, p) \mid T \in [1, T_{max}], p \in [c, p_{max}] \} \), which is so-called compact set in a plane. It is obvious that function \( Z_1 \) is continuous even on this rectangle.

In addition, it follows from so-called Weierstrass theorem (see differential calculus of multivariable functions – e.g. Došlá and Kuben (2012)) that function \( Z_1 \) reaches both its minimum and its maximum value on rectangle \( O \). The function may reach these values either in points of local extremes (inside the rectangle), or on its borders.

Since it holds true that

\[ \frac{\partial^2 Z_1(T,p)}{\partial T^2} = -\frac{2s}{T^3} - Ha - apI_f < 0, \]  

(4)

function \( Z_1 \) is a concave function on rectangle \( O \), its potential extreme is the maximum of the function.

**Case in which the goods are sold after the end of the credit period**

If \( T \geq m \), at time \( m \) the retailer needs to take out credit and the interest payable will be \( \alpha T^2 (T - m)^2 I_c / 2 \).

Hence, the annual interest payable will be \( \alpha T^2 (T - m)^2 I_c / 2 \) (Goyal, 1985).

Nevertheless, the retailer may earn interest on revenue gained before time \( m \) and deposited in an account. The interest earned is \( \alpha m^2 I_d / 2 \), and the interest earned per year is \( \alpha m^2 I_d / 2T \).

Hence, the total annual profit is

\[ Z_2(T,p) = \frac{paT}{2} - \frac{s}{T} - \frac{cT}{2} - \frac{HaT^2}{2} - \frac{ap(T-m)^2 I_c}{2T} + \frac{\alpha m^2 I_d}{2T}. \]  

(5)

At the same time it holds true that

\[ \frac{\partial^2 Z_2(T,p)}{\partial T^2} = -\frac{2s}{T^3} - Ha - apI_f < 0. \]  

(6)

Therefore, under above mentioned conditions, function \( Z_2(T,p) \) is a concave function and its extreme is the maximum of function \( Z_2 \).

**Illustrative example**

Let's assume the following values of a fictitious business:

- \( H = 0.5 \) (units per year), \( I_c = 0.02 \) (currency unit per year), \( I_d = 0.005 \) (currency units per year), \( c = 0.1 \) (currency units), \( s = 5 \) (currency units per item), \( \alpha = 100000 \), \( \beta = 1.6 \), \( m = 10/365 \) (year).

The task is to set the optimal price of goods per item and the maximum selling period generating profit for the retailer.

In compliance with the previous text, we assume that \( p > c \) and the goods cannot be sold until the first day, hence \( T > 0 \).

**If the goods are sold before the end of the credit period \((T \leq m)\)**

If the retailer assumes that goods will be sold under given conditions, he maximizes his profit only if \( T = 0.027 \) and \( p = 1.6 \). The graph in Fig. 1 shows a plane which demonstrates all possible situations which may occur under the given conditions as long as the retailer does not incur loss.

**If the goods are sold after the end of the credit period \((T > m)\)**

If the retailer knows that he will not be able to meet the deadline, price \( p \) is determined by maximizing function \( Z_2 \), because the retailer has to pay interest payable starting at moment \( T > m \). The retailer can maximize his profit if price \( p = 2.59 \) and the credit is paid off at time \( T = 19.33 \).

**If the goods are unexpected sold after the end of the credit period**

If the retailer's original assumption is that the goods will be sold "in time" but the assumption is not met, price \( p \) is determined in the same way as in the previous case, but, starting at time \( T > m \), the retailer also needs to pay interest payable. In such a case, however, retailer's costs exceed income at time \( T = 31.3 \). The option of making profit is illustrated by the graph in Fig. 2.

The impact of changes in value of some parameters on the model's behaviour.

Thanks to exactness of the model and availability of appropriate software, we are able to assess the impact of potential change in external factors. In order to demonstrate the changes, we have chosen...
a change in interest rate $I_c$ and a change in purchasing price $c$.

**The impact of changes in parameter $I_c$ on the selling period**

An increase in $I_c$ value means that the retailer needs to sell the goods considerably faster when the maturity date was not met. If $I_c = 0.2$ (while keeping all other parameters at the same value), the critical period for selling goods is $T = 3.13$ when $p = 1.6$. Graphical illustration can be found in Fig. 3. A decrease in $I_c$ on the contrary, causes an extension of the period during which the goods can be sold without putting the retailer in risk of suffering loss almost absurd $T = 312.5$.

**The impact of change in parameter $c$ on the selling period**

A decrease in the purchase price $c$ to $c = 0.01$ leads to setting up an optimum price $p = 1.36$ on condition of early settlement and enables the retailer to postpone the sale of goods after the maturity date till time $T = 2.56$. However, if $c$ rises to 1.0 (while keeping price $p = 3.99$), the goods can be sold at profit till time $T = 5.16$. These changes can be observed if all the other parameters are kept at original values.

**CONCLUSION**

The paper introduced a model calculating the optimum price and the maximum term of payments in case of credit on goods in which the retailer is granted a permissible delay in payments by the supplier. The paper further analyzed a situation in which the retailer sell all the goods in time, and a situation in which the deadline was not met.

Theoretical results were demonstrated by an illustrative example, which also displayed the specific results graphically. An impact of changes in the values of some parameters is demonstrated in this example. A change in the level of parameter $I_c$ influences significantly the period that the trader has to sell the goods in order to make profit. A change in value of the purchase price (parameter $c$) affects both the selling price of goods and the permissible time of sale (the replenishment interval).

The model suggested in the paper can be expanded in the future. One option is considering generalization of the model allowing for the lack of goods, bulk discounts, etc.

**SUMMARY**

Traditional inventory models assume that a retailer pays for the goods the moment they are received. Nowadays, however, it is becoming a common practice that a supplier offers a retailer the option to pay for the goods with a certain delay. The retailer may, before the end of the period, sell the goods, accumulate revenue and earn interest. If the retailer is not able to meet his obligations within the deadline, he is charged an interest. In other words – a supplier provides a retailer with an interest-free credit for a period set in the contract. As there are many ways of determining the interest payable and the interest earned, Goyal’s model was used for the sake of simplification throughout the paper.

The papers aims to set up a mathematical model which enables a retailer to fix (based on the knowledge of certain parameters) the optimum selling price per an item of goods and to determine the maximum interval during which goods can be sold at a profit. The model is based on the assumption of time-dependent demand and is developed for non-deteriorating goods. Further assumption is that the inventory is depleted only by demand.

The scientific aim is to verify if such an optimizing problem can be solved. The authors of the paper used analytic and synthetic methods and the method of mathematical analysis (differential calculus of multivariable functions, solution of ordinary differential equations).
The new model is demonstrated in the paper by a specific problem as well as graphical solution. All calculations were carried out using Maple system, which is used as mathematical software mainly because of its capability to solve problems in a symbolic way. The paper further analysed a situation in which the retailer sell all the goods in time, and a situation in which the due period. Thanks to exactness of the model and availability of suitable software, we are able to assess the impact of a potential change in external factors. Theoretical results are demonstrated by an illustrative example, presenting concrete results graphically.

The model suggested in the paper can be expanded in the future. One option is generalization of the model, allowing for the lack of goods, bulk discounts, etc.

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Address

Ing. Martina Janková, Mgr. Veronika Novotná, Ing. Tereza Varyšová, Institute of Informatics, Brno University of Technology FBM, Kolejní 2906/4, 612 00 Brno, Czech Republic, e-mail: jankova@fbm.vutbr.cz, novotna@fbm.vutbr.cz, varysova@fbm.vutbr.cz