SUMMATION COMPARISON THEOREMS FOR HALF-LINEAR SECOND ORDER DIFFERENCE EQUATIONS ON FINITE INTERVAL

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Abstract


In the paper, new comparison theorems for the half-linear difference equation

$$\Delta \left( R_k \Phi(\Delta z_k) \right) + C_k \Phi(z_{k-1}) = 0, \quad \Phi(u) = |u|^{p-2}u, \quad p > 1,$$

are derived. We show that if a solution of this equation has a generalized zero on the discrete interval $[a,b]$, then the same holds for a solution of its majorant. The main tool used in the paper is the variational technique which relates nonexistence of a solution with a generalized zero with nonnegativity of the $p$-degree functional defined on the suitable class of admissible functions.

difference equation, second order, focal point, half-linear equation, $p$-degree functional, free end point

1. INTRODUCTION

Consider the second order half-linear difference equation

$$\Delta \left( R_k \Phi(\Delta x_k) \right) + C_k \Phi(x_{k-1}) = 0,$$  \hspace{1cm} (1)

where $\Delta$ is the forward difference operator, $(C_k)$, $(R_k)$ are real sequences, $R_k \neq 0$ for $k = 0,\ldots,n+1$, and $\Phi(u) = |u|^{p-2}u$, $p > 1$, is a power type nonlinearity. The study of equation (1) has been initiated in Řehák (2001) and the most important results are summarized in Chapter 8 of the monograph Došlý, Řehák (2005).

Despite the lack of linearity, a constant multiple of any solution of (1) is also a solution and equation (1) has one half of linearity properties. It is well known that there is a close similarity between equation (1) and the linear second order difference equation.

In particular, many results from oscillation theory of second order linear difference equations can be extended to (1). These oscillation and nonoscillation results are frequently based on a comparison of two equations on the infinite interval. The aim of this paper is to derive comparison theorems which compare (1) with another half-linear difference equation

$$\Delta \left( r_k \Phi(\Delta y_k) \right) + c_k \Phi(y_{k-1}) = 0$$  \hspace{1cm} (2)

on a finite interval.

First, let us recall the definition of a generalized zero, which is (from the point of view of the Sturmian comparison theory) a natural replacement for zeros of solutions to differential equations. Remark that, unless stated explicitly otherwise, under the interval $[m,n]$ we actually mean the
Definition 1. The interval $[m, m + 1]$ is said to contain a generalized zero of a solution $x = (x_k)$ of Eq. (I) if $x_k = 0$ and $R_{xx}x_{m+1} \leq 0$.

It is well known, see for example Řehák (2001) or Došlý, Řehák (2005), that equation (1) tends to have more generalized zeros than (2), if the inequalities $R_1 \leq r$, and $C_i \geq c_i$ are satisfied. In contrast to the pointwise comparison we formulate our results more generally in terms of sums of the coefficients $C_i$ and $c_i$. Our aim is to derive a discrete version of the following theorem due to Leighton.

Theorem A (Leighton (1983), Theorem 1.1). Let $p(t)$ and $q(t)$ be piecewise continuous on $[a, f]$ with $q(x) \geq 0$ there, and suppose that
\[
\int_a^b p(t) \, dt \leq \int_a^b q(t) \, dt, \quad a \leq t \leq f; \quad p(x) \neq q(x)
\]
holds. If equation
\[
u'' + q(t)u = 0
\]
has a solution $u(t)$ with the property that $u'(a) = u(f) = 0$, $u(x) > 0$ on the interval of real numbers $[a, f]$, a solution $v(t)$ of
\[
u'' + q(t)v = 0
\]
with $v'(a) > 0, v'(a) < 0$ must have a zero on the interval of real numbers $(a, f]$.

Another aspect which makes our results different from those published in the literature is that similarly as in Theorem A we compare two solutions which do not vanish at the left end point of the interval. As far as the author knows, the results are new even for the linear difference equation.

The main tool used in the paper is the variational technique which relates equation (1) and the corresponding discrete scalar $p$-degree functional
\[
J(\eta) = \|A\eta\|^p + \sum_{k=0}^{n-1} (R_k |\Delta \eta_k| - C_k |\eta_k|)^p. \quad A \in \mathbb{R}
\]
defined on the class of nontrivial sequences $\{\eta_k\}_{k=0}^{n-1}$ such that $n_{n+1} = 0$. Note that since we aim to compare the solutions which do not vanish at the left end point of the interval, we drop the usual requirement $n_k = 0$ from the definition of the admissible sequences for functional $J$ and also include the term $A|\eta_k|^p$.

The relationship between the half-linear difference equation and the $p$-degree functional is established in the following theorem.

Theorem B (Mařík (2000), Theorem 1). The following statements are equivalent:

i) The solution $x = (x_k)$ of Eq. (I) given by $R_k \Phi(\Delta x_k) = A$ has no generalized zero on $[0, n + 1]$.

ii) Functional (3) is positive definite on the class of nontrivial sequences $\eta = \{\eta_k\}^{n-1}_{k=0}$ satisfying $n_{n+1} = 0$.

The following result allows to compare two solutions of two different equations and it is an immediate consequence of Theorem B. The crucial aspect of the proof of this theorem lies in the fact that the functional $J$ vanishes for the sequence which solves equation (1) and satisfies initial condition closely connected with the value $a$.

Theorem C (Leighton type comparison theorem, Mařík (2000), Corollary 1). Let $y = (y_k)$ be a solution of Eq. (2), such that $y_{n+1} = 0 = y_0$, and $a := r_k \Phi(\Delta y_k) \overline{y_k}$. Let $A$ be such that
\[
V(y) = (A - a)|y_0|^p + \sum_{k=0}^{n-1} (R_k - r_k)|\Delta y_k| - (C_k - c_k)|y_{k+1}|^p \leq 0.
\]

Then the solution $x = (x_k)$ of Eq. (1) given by $R_k \Phi(\Delta x_k) = A$ has a generalized zero on $(0, n + 1)$, i.e., there exists $i \in [0, n]$ such that $x_i = 0$ and $R_{xx}x_{n+1} \leq 0$ holds.

2. Main results

This section contains the main results of the paper. In the following theorem we prove that if the solution of half-linear differential equation (2) vanishes at the point $n + 1$, then the solution of the equation (1) with a sufficiently large coefficient $C_i$ has a generalized zero on $[0, n + 1]$, if the initial difference is negative and not too large. However, the words “sufficiently large” are here in the integral sense as (4) shows. Hence the inequality $C_i \geq c_i$ is not necessary for all $i$.

Note the technical assumption on nonnegativity of $c_i$ which assures that the solution which starts with positive initial value and nonpositive initial difference is nonincreasing.

Theorem 1. Let $r_k > 0$ on $[0, n]$, $c_k \geq 0$ on $[0, n - 1]$, $c_n > 0$. Let $y$ be a solution of (2) on $[0, n + 1]$ such that $y_n > y_1 > 0, y_0 > 0$ on $[0, n]$ and $y_{n+1} = 0$. Denote $a := r_k \Phi(\Delta y_k) \overline{y_k}$. Let $R_k \leq r_k$ on $[0, n]$, $C_k \geq c_k$, $A \leq a$ and
\[
\left|\begin{array}{l}
\left|y_0\right| + \sum_{k=0}^{n-1} (C_k - c_k) \\
\left|y_1\right|
\end{array}\right| (A - a) + \sum_{k=0}^{n-1} (C_k - c_k) \geq 0
\]
for $k \in [0, n - 1]$. Then the solution $z = (z_k)$ of (1) given by the conditions $z_0 > 0$, $A = R_k \Phi(\Delta y_k)\overline{z_k}$ has a generalized zero on $(0, n + 1)$, i.e., there exists $i \in [0, n]$ such that $z_i = 0$ and $R_{zz}z_{n+1} \leq 0$.

Proof. From $R_k \leq r_k$ we get
\[
V(y) \leq (A - a)|y_0|^p + \sum_{k=0}^{n-1} (C_k - c_k)|y_{k+1}|^p
\]
Further, from (2) it follows
\[
\Phi(\Delta y_{n+1}) = \frac{r_k}{r_k} \Phi(\Delta y_k) - c_k \overline{y_k}
\]
for $k \in [0, n - 1]$ and
for $k \in [0, n-1]$. Hence $|y_{k-1}^n|$ is decreasing on $[0, n]$. Clearly there exists $\epsilon \in \mathbb{R}, \epsilon > 0$, such that the intervals of real numbers $I_k := (|y_{k-1}^n| - \epsilon, |y_{k-1}^n| + \epsilon) \subseteq \mathbb{R}$, $k \in [1, n]$, satisfy $I_k \cap I_j = \emptyset$ for $j \neq k$. In each $I_k$ let us choose $\alpha_k, \alpha_k \in I_k \cap \mathbb{Q}$, such that

$$(c_k - C_k)\|y_{k-1}^n\| \leq (c_k - C_k)\alpha_k$$

for $k \in [0, n-1]$. (5)

Denote by $\beta$ the least common multiple of denominators of $\alpha_k$. Then the numbers $\beta_k$, defined by $\beta_k = \beta \alpha_k$ form a decreasing sequence for $k \in [1, n]$ and $\beta_k \in \mathbb{N}$. Combining these computations with $y_{n+1} = 0$ we obtain

$$V(y) \leq (A - a)\|y\|^p + \sum_{k=0}^{n} (c_k - C_k)\|y_{k-1}^n\|^p$$

$$\leq (A - a)\|y\|^p + \frac{1}{\beta_k} \sum_{k=0}^{n} (c_k - C_k)$$

Changing the order of summation we get

$$V(y) \leq (A - a)\|y\|^p + \frac{1}{\beta_k} \sum_{i=0}^{n} (c_i - C_i).$$

where $\gamma$ is a well defined number from the discrete interval $[0, n]$. More precisely, $\gamma_k$ denotes how many times the number $k$ appears in the double sum $\sum_{k=0}^{n} \sum_{i=0}^{n} k^i$. By (4), we obtain

$$V(y) \leq (A - a)\|y\|^p + \frac{\beta_k}{\beta_k} \sum_{k=0}^{n} (c_k - C_k).$$

Since (5) and $C_k \geq C_i$ imply $a_k \leq |y_{k-1}^n|$, we have $V(y) \leq 0$. Now the statement follows from Theorem C.

There is a variant of Theorem 1 which is based on the nonnegativity of slightly different sum than (4). Namely, the coefficient $c_k$ has the weight $\beta_k \alpha_k$ in this sum. To derive this modification of Theorem 1 we need the following Lemma 1. This lemma is proved in Marík (2000), Corollary 3, as a corollary of Theorem B. However, the original version contains some misprints and for this reason we restate this lemma with a shorter proof than the proof presented in Marík (2000).

**Lemma 1.** Let $y = (y_k)$ be a solution of Eq. (2) on $[0, n-1]$, such that $y_n = 0$ and $a = \delta \Phi(A_{y_n})$. Let $A$ be such that

$$\varphi(y) = (A - R_k \alpha_k)\|y\|^p + \sum_{k=0}^{n} (A - R_k \alpha_k)\Phi(A_{y_k}) \Phi(y_{k+1}) + (c_k - R_k \alpha_k)\|y_{k-1}^n\|^p \leq 0.$$

Then the solution $z = (z_k)$ of Eq. (1) given by

$$\varphi(z) = (R_k \alpha_k) \Phi(A_{y_k}) \Phi(y_{k+1}) + (c_k - R_k \alpha_k)\|y_{k-1}^n\|^p \leq 0.$$

Proof. Let $y = (y_k)$ be a solution of (2) on $[0, n-1]$ which satisfies $y_{n+1} = 0$ and $a = \delta \Phi(A_{y_n})$. Then

$$L(y) = \sum_{k=0}^{n} (A - R_k \alpha_k)\Phi(A_{y_k}) \Phi(y_{k+1}) + (c_k - R_k \alpha_k)\|y_{k-1}^n\|^p.$$

Therefore in view of (6) and $y_{n+1} = 0$, clearly

$$J(y) = \sum_{k=0}^{n} \varphi(y_k) \|y\|^p + \sum_{k=0}^{n} \varphi(y_{k+1}) \|y\|^p$$

and the statement follows from Theorem B.

**Theorem 2.** Let $\{y_k\}_{k=0}^{n}$, $\{\alpha_k\}_{k=0}^{n}$, $\{C_k\}_{k=0}^{n}$, $\{C_i\}_{i=0}^{n}$, $\{R_k\}_{k=0}^{n}$, $\{R_k \alpha_k\}_{k=0}^{n}$, $\{\beta_k\}_{k=0}^{n}$, $\{\gamma_k\}_{k=0}^{n}$ be a solution of (2) on $[0, n-1]$, such that $y_0 \geq y_1 \geq \cdots \geq y_n = 0$ and $a = \delta \Phi(A_{y_n})$ has a generalized zero

$$\sum_{k=0}^{n} (R_k \alpha_k) \Phi(A_{y_k}) \Phi(y_{k+1}) + \sum_{k=0}^{n} (R_k \alpha_k) \|y_{k-1}^n\|^p \leq 0.$$
on $[0, n + 1]$, i.e., there exists $i \in [0, n]$ such that $x_i \neq 0$ and $R_i x_{i+1} \leq 0$.

**Proof.** From the assumption $\Delta_k R \leq 0$ we get

$$V(y) \leq \left( A - \frac{R_k}{\tau_k} \right) |y|^p - \sum_{i=k}^{n} \left( \frac{C_i - R_i}{\tau_{i+1}} \right) |y_{i+1}|^p$$

The remaining part of the proof is essentially similar to the proof of Theorem 1 where we replace $V(y), a$ and $c_k$ by $\tilde{V}(y), \frac{R_0}{\tau_0} a$ and $\frac{R_n}{\tau_n} c_n$, respectively.

**SUMMARY**

The classical results in the comparison theory of half-linear differential and difference equations deal with the generalized zeros of solutions which vanish at the left end point of the interval. Focal points, i.e. generalized zeros of solutions which start with zero difference, can be considered as a natural continuation of this research. The results presented in this paper include focal points if we choose $A = a = 0$ in Theorems 1 and 2.

Another companion of the conjugate point and the focal point is also the so called coupled point, the point associated with functional defined on another class of admissible functions, such as functional with free end points. Theory of discrete coupled points has been introduced in a series of papers by Hilscher and Zeidan, see Hilscher, Zeidan (2002, 2004, 2005) and the references therein. The possible extension of coupled point to half-linear equation and possibility to formulate comparison theorems in terms of coupled points is still an open question and a subject of the current research.

Further, there are results from the theory of differential equations, which allow to study nonoscillatory half-linear differential equations as a perturbation of another half-linear equation. This technique has been started in the paper Elbert, Schneider (2000) and extended in a series of papers by Došlý and coauthors. Among others, it has been shown that this method extends to difference, see e.g. Došlý and Fišnarová (2008, 2009), and can be formulated in variational setting, see Došlý and Fišnarová (2011).

We hope that developing similar method for functional (3) instead of functional with zero end points opens the door to future extensions.

**REFERENCES**


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